Operational Quantum Cosmology: Consistent Relational Histories

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Abstract

We present Consistent Relational Histories (CRH), a framework that integrates the Consistent Histories (CH) probability calculus with relational observables and quantum reference-frame (QRF) transformations, aimed at describing closed quantum systems such as the early universe. CRH does not modify the dynamics of canonical or covariant quantum gravity; rather, it provides the operational structure needed to formulate questions, construct histories, and assign probabilities in the absence of external observers or an absolute time.

Histories are defined using relational projectors that condition subsystem properties on internal clocks or reference systems. A relational Heisenberg evolution arises from a uniquely defined generator extracted from the Hamiltonian constraint, ensuring gauge invariance under admissible clock–system decompositions. QRF transformations are required to satisfy precise compatibility criteria so that consistent families, decoherence functionals, and probability assignments remain meaningful across frames.

The resulting formalism predicts frame-dependent deviations whenever internal clocks have quantum fluctuations or become entangled with the systems whose histories they label. We demonstrate this with an explicit oscillator–qubit model and provide order-of-magnitude estimates for minisuperspace cosmologies, showing that deviations vanish in classical limits but may become significant near quantum–gravitational regimes. CRH thus serves not as a new theory of quantum gravity, but as a coherent probabilistic and relational framework that renders histories, frame changes, and empirical predictions operationally well-defined in closed-universe settings. CRH's contribution is orthogonal: it furnishes the operational and probabilistic structure required to formulate relational questions and assign consistent probabilities for any quantum gravity theory. Additionally, CRH can be interpreted as a concrete, quantum, definite-order realization of a broader Timeless Operational Relational Network (TORN) framework, providing the structural setting in which explicit calculations and physical predictions can be carried out.

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1 Introduction

A closed quantum universe lacks external structures that ordinarily anchor the formulation of quantum theory. There is no classical background time, no external observer, and no preferred reference frame. Any meaningful physical statement must therefore be expressed internally, as a relation among the degrees of freedom of the universe itself. While several existing frameworks— such as the Consistent Histories (CH) formalism, Relational Quantum Mechanics (RQM), and the theory of Quantum Reference Frames (QRF)—each illuminate aspects of this relational perspective, none by itself supplies a complete operational prescription for asking and answering probabilistic questions in a background-independent context.

The purpose of this paper is to develop a unified, operational framework—called *Consistent Relational Histories* (CRH)—that integrates the probability assignment machinery of CH with the relational observables of RQM and the transformation theory of QRFs. CRH is not a modification of the dynamics of quantum gravity; rather, it provides the probabilistic and semantic layer needed to formulate questions, construct histories, and compare descriptions defined relative to different internal clocks or reference systems.

The framework begins by replacing absolute-time projectors with *relational projectors* that condition the properties of a subsystem on a chosen internal clock or reference degree of freedom. These relational observables generate histories that are meaningful in the absence of external temporal or observational structure. A relational Heisenberg evolution follows naturally from the Hamiltonian constraint, leading to a uniquely defined generator that is invariant under admissible redefinitions of the clock–system split. This construction restores a notion of physical time evolution without introducing an external parameter.

Because different internal clocks or reference systems correspond to different choices of conditionalization, CRH requires a principled way of translating histories across perspectives. To this end, we introduce precise compatibility criteria that determine when a QRF transformation preserves the algebra of relational observables, maps valid histories to valid histories, and maintains CH consistency. Only such transformations are regarded as describing the same physical question from different viewpoints; in all other cases, the resulting probabilities may legitimately differ. This distinction clarifies when frame-dependence is a gauge redundancy and when it corresponds to inequivalent relational descriptions.

The combination of relational conditioning, consistent histories, and compatibility-restricted QRF transformations yields new empirical possibilities. Whenever internal clocks possess quantum fluctuations or become entangled with the systems whose properties they label, the decoherence functional and associated probabilities need not be invariant across relational frames. We illustrate this phenomenon with a nontrivial oscillator–qubit model and provide order-of-magnitude estimates for minisuperspace cosmologies, showing that deviations vanish in classical limits while remaining relevant in quantum–gravitational regimes.

In summary, CRH provides a coherent operational (Table 1) framework for formulating probabilities, comparing perspectives, and extracting relational dynamics in closed-universe settings. It complements existing approaches to canonical, covariant, and loop quantum gravity by supplying the structure needed to pose and evaluate physically meaningful questions in the absence of external time or observers.

1.1 Related Work

Several frameworks address the challenge of formulating quantum theory for a closed system without external time or observers.

Consistent Histories (CH). CH assigns probabilities to sequences of quantum events via the decoherence functional [11, 16, 7]. While powerful, CH does not provide rules for constructing histories when times are themselves quantum degrees of freedom. CRH builds directly on CH but replaces absolute-time projectors with relational ones.

Relational Quantum Mechanics (RQM). RQM [22, 18] emphasizes that physical properties refer to relations between systems. It motivates the relational semantics used in CRH but does not supply a probability theory for relational sequences.

Quantum Reference Frames (QRF). QRF transformations [9, 13] show how descriptions of physical processes change when the reference system is quantum. CRH incorporates QRFs but restricts attention to transformations preserving relational observables, valid histories, and CH consistency.

Quantum Gravity Contexts. Canonical and covariant quantum gravity encode dynamics through Hamiltonian or diffeomorphism constraints [14, 17, 1], but do not offer an operational probability framework. Relational dynamics à la Page–Wootters [19] or conditional observables address time relationally but lack a unified history-based semantics. CRH complements these approaches by supplying the needed probabilistic structure.

2 Background: Consistent Histories, Relational Quantum Theory, and Quantum Reference Frames

The Consistent Histories (CH) formulation of quantum mechanics was developed as an observer-independent framework in which probabilities can be assigned to sequences of quantum events in a closed system, such as the universe. Although it was not designed with explicitly “relational” language, its logical architecture is deeply compatible with more recent relational approaches, including Relational Quantum Mechanics (RQM)

Feature	**Operationalism**	**Realism**	**Pragmatism**	**Empiricism**
What is the quantum state?	A tool encoding probabilities for outcomes of operations; not a description of reality.	A physically real entity (ontic state) describing the world.	A guide for rational expectations; agent-centered.	A compact summary of observable correlations; not ontological.
Ontology	Avoided; theory is about procedures and statistics.	World has observer-independent properties or structures.	Ontology is secondary to usefulness; meaning = practical success.	Only observable phenomena count as legitimate referents.
View of Measurement	Defined operationally as preparation, interaction, measurement acts.	Explained via underlying ontology (e.g., branches, particles, collapses).	A decision-making update for an agent.	Boundary of what can be known; classical description required.
Role of the Observer	An external agent performing operations.	An element within or outside the ontic world depending on interpretation.	An active agent updating beliefs.	A classical observer registering outcomes.
Use in Cosmology	Problematic (no external experimentalist).	Naturally suited (universe has ontic state).	Possible but requires redefining the "agent."	Difficult (no observation-from-outside possible).
Philosophical Ancestry	Bridgman, operational definitions; quantum information theory.	Einstein, Bell, realist metaphysics; Bohm, Everett, GRW.	James, Dewey; QBism; pragmatist philosophy of science.	Hume, Mach; Bohr's emphasis on phenomena.
Strengths	Clarity, minimal assumptions; directly tied to experiments.	Explanatory depth, ontological completeness.	Resolves many paradoxes by reframing questions.	Avoids unverifiable metaphysics; emphasizes observable structure.
Weaknesses	Avoids the "what exists?" question.	Conflicts with no-go theorems; heavy metaphysical load.	Less suitable for universal, closed-system physics.	Hard to apply to cosmology; unclear referent of theory.

Table 1: Comparison of Operationalism, Realism, Pragmatism, and Empiricism in Quantum Foundations.

in the sense of Rovelli, quantum reference frame (QRF) formalisms, and background-independent approaches to quantum gravity and quantum cosmology.

In this section we spell out these connections. At a high level, one can summarize the relation as follows:

> *Consistent Histories provides a perspectival* probabilistic logic *for quantum theory, while relational and QRF approaches provide a perspectival* ontology and dynamics. *Both reject a single, absolute, frame-independent description of quantum reality.*

2.1 Consistent Histories: Frameworks and the Single-Framework Rule

In the CH approach, a *history* of a closed quantum system is a temporally ordered sequence of projectors on the system's Hilbert space,

$$\alpha = \big(P_1(t_1), P_2(t_2), \ldots, P_n(t_n)\big), \tag{1}$$

where each $P_k(t_k)$ is a Heisenberg-picture projector representing a quantum property (e.g. a range of positions, spins, or coarse-grained field configurations) at time t_k. Given an initial density operator ρ, one defines the *class operator*

$$C_\alpha = P_n(t_n) \cdots P_2(t_2) P_1(t_1), \tag{2}$$

and the *decoherence functional*

$$D(\alpha, \alpha') = \mathrm{Tr}\big(C_\alpha \, \rho \, C_{\alpha'}^\dagger\big). \tag{3}$$

A set $\mathcal{F} = \{\alpha\}$ of alternative histories is called *(medium) consistent* if the off-diagonal elements of the decoherence functional vanish,

$$D(\alpha, \alpha') = 0 \quad forall \alpha \neq \alpha' \in \mathcal{F}. \tag{4}$$

In this case, the diagonal values

$$p(\alpha) = D(\alpha, \alpha) \tag{5}$$

can be interpreted as classical probabilities satisfying the Kolmogorov sum rules. Such a set $\mathcal{F}$ is a *consistent family* or *framework*.

Griffiths emphasizes that quantum reasoning is only meaningful *within* a single consistent family. This is the *Single-Framework Rule*:

> It is not permissible to combine propositions or probability assignments originating from two incompatible consistent families (e.g. defined by projectors onto noncommuting observables); doing so leads to logical contradictions and "paradoxes" of the usual textbook kind.

Thus, in CH there is no unique, globally valid description of a quantum system. Rather, there are multiple, mutually incompatible but individually self-consistent descriptions, each realized as a consistent family of histories.

2.2 Relational Quantum Mechanics: Facts Relative to Systems

Relational Quantum Mechanics (RQM) starts from a different angle, but arrives at a structurally similar conclusion. RQM proposes that quantum states and properties are meaningful only *relative to another physical system* (often called an "observer," though not necessarily conscious). There is no observer-free, God's-eye quantum state of the universe; instead, there are many relational states $|\psi\rangle_{S|O}$ describing a system S relative to a system O.

The central relational principle is:

> A quantum event (e.g. the value of an observable) is a fact only *relative* to a system with which the measured system has interacted.

Different observers may consistently assign different states and even different sequences of events, provided they respect the rules of quantum theory when they later interact and compare records.

Comparing this to CH, we see a strong structural resonance:

- In RQM, *facts* are defined relative to an observer-system relation.
- In CH, *properties and probabilities* are defined relative to a chosen consistent family (framework).

Both deny the existence of a single, absolute, observer-independent history of the world accessible to all descriptions simultaneously. The difference is largely one of emphasis: RQM foregrounds the ontology of relational facts, while CH foregrounds the logic of probabilistic reasoning within a framework. One can heuristically summarize:

$$\text{RQM} \simeq \textit{perspectival ontology}, \qquad \text{CH} \simeq \textit{perspectival probabilistic logic.}$$

2.3 Quantum Reference Frames: Transforming Between Perspectives

Recent work on quantum reference frames (QRFs) takes the relational idea one step further by treating reference frames themselves as quantum systems, and by explicitly constructing unitary transformations between different "quantum perspectives." The basic statement is that physical states and observables take different forms depending on the quantum reference frame, and that there is no preferred, absolute frame.

From a structural perspective, a QRF plays a role analogous to a CH framework:

1. A QRF defines a complete set of degrees of freedom and observables with respect to which descriptions are made (e.g. "the center-of-mass frame of system A" versus "the frame of system B").
2. A CH framework defines a complete set of projectors and histories with respect to which probabilities are assigned.

In both cases, the description is *frame-dependent* and cannot be arbitrarily combined with another incompatible description without care.

There is, however, a key difference:

- QRF formalisms typically provide explicit unitary maps between different reference frames, allowing one to transform physical states from one quantum perspective to another.
- In CH, incompatible frameworks are not related by a simple transformation; they are analogous to incompatible bases in Hilbert space, and the Single-Framework Rule forbids mixing them in a single logical argument.

Despite this difference, both CH and QRFs embody a common conceptual core: physical statements are *perspective-dependent*, and consistency must be enforced within a well-defined perspective (framework or reference frame).

2.4 Quantum Cosmology and Background-Independent Quantum Gravity

The CH formalism was partly motivated by the need to apply quantum mechanics to *closed systems*, such as the early universe, where there is no external classical measuring apparatus. This motivation is shared by path-integral and decoherent-histories approaches to quantum cosmology, notably the work of Gell-Mann and Hartle. In such settings, quantum gravity and cosmology are often formulated without a fixed background spacetime and without a global time parameter (e.g. the Wheeler–DeWitt framework; loop quantum gravity and related approaches).

Background-independent quantum gravity exhibits several features that align naturally with CH:

- **No preferred time:** Time is not an external parameter but is extracted from correlations among dynamical degrees of freedom. In CH, histories are sequences of events labeled by an ordering parameter, but the formalism itself does not rely on an external classical time; in quantum cosmology, one often works directly with histories of geometric and matter configurations.
- **No fixed background:** Spacetime geometry is dynamical and relational. CH allows one to define coarse-grained histories of geometry and matter and to assign probabilities to entire spacetime histories of the universe, rather than to instantaneous states on a fixed background.
- **Multiple coarse-grainings:** Different coarse-grainings of the underlying degrees of freedom correspond to different consistent families of histories. Each gives a different, but individually valid, "narrative" of the universe. This mirrors the idea that different choices of relational variables or different spacetime decompositions in quantum gravity can yield different but physically equivalent descriptions.

In this sense, CH provides a natural probabilistic superstructure for background-independent, relational theories of quantum gravity: it supplies the rulebook for when and how it is legitimate to assign probabilities to entire histories of the universe, given that there is no external observer or measurement apparatus.

2.5 Summary: A Common Relational Logic

To summarize, the connections between Consistent Histories and relational approaches to quantum theory and quantum gravity can be captured as follows:

- CH replaces a single, absolute quantum history with multiple, mutually incompatible but individually consistent families of histories, and restricts reasoning to a single framework at a time (Single-Framework Rule).

- RQM replaces an absolute quantum state with many relational states, each valid only relative to a given system; different observers can assign different facts without contradiction, provided their interactions are properly treated.
- QRF approaches replace a preferred reference frame with a multiplicity of quantum frames, linked (when possible) by explicit unitary transformations.
- Background-independent quantum gravity replaces a fixed spacetime arena with a network of dynamical, relational degrees of freedom, for which CH can provide a natural, observer-free probabilistic interpretation at the level of entire cosmological histories.

Viewed this way, Consistent Histories is not merely another "interpretation" of quantum mechanics (see Table 2), but a conceptual bridge: it delivers a rigorous, logically consistent, observer-free probabilistic framework that dovetails with the relational and frame-dependent structures emerging in modern quantum gravity and quantum information theory.

3 Consistent Relational Histories in Quantum Reference Frames for Quantum Cosmology

In this section we propose a unified framework for quantum cosmology that synthesizes three influential ideas: (i) the Consistent Histories (CH) formulation of quantum mechanics [11, 12, 7], (ii) relational quantum theory in the sense of Rovelli [22], and (iii) the recent quantum reference frame (QRF) program [9, 24]. The resulting structure, which we call *Consistent Relational Histories* (CRH), is well-suited for closed-universe quantum cosmology, where there is no external observer, no fixed background time, and no preferred classical reference frame.

The central idea of CRH is straightforward: a "history" is defined as a sequence of *relational quantum events* (properties of one subsystem relative to another), and probabilities are assigned only within *consistent* families of such relational histories, each defined in a particular quantum reference frame. Different quantum reference frames are related—when possible—by unitary transformations, and a generalized version of the Single-Framework Rule ensures logical consistency.

3.1 Kinematics of a Closed Universe

Let $\mathcal{H}$ denote the Hilbert space of the universe, factored into dynamical subsystems

$$\mathcal{H} = \mathcal{H}_G \otimes \mathcal{H}_M \otimes \mathcal{H}_C \otimes \mathcal{H}_O \otimes \cdots, \tag{6}$$

where $\mathcal{H}_G$ contains geometric (gravitational) degrees of freedom, $\mathcal{H}_M$ matter fields, $\mathcal{H}_C$ an "internal clock" variable, and $\mathcal{H}_O$ an observer or memory subsystem (not necessarily conscious). States of the universe satisfy a Hamiltonian constraint

$$\widehat{H}\,|\Psi\rangle = 0, \tag{7}$$

as in Wheeler–DeWitt or loop quantum cosmology.

3.2 Relational Projectors and Relational Histories

Following Rovelli, properties of a subsystem are defined only *relative to another subsystem.* A relational projector is denoted

$$P_k^{(S|O)}(\tau_k), \tag{8}$$

meaning: subsystem S has property A_k *when* observer/clock subsystem O records value τ_k.

A *relational history* in the reference frame of O is thus

$$\alpha^{(O)} = \big(P_1^{(S_1|O)}(\tau_1), P_2^{(S_2|O)}(\tau_2), \ldots, P_n^{(S_n|O)}(\tau_n)\big). \tag{9}$$

The corresponding class operator is

$$C_\alpha^{(O)} = P_n^{(S_n|O)}(\tau_n) \cdots P_1^{(S_1|O)}(\tau_1). \tag{10}$$

Aspect	**Consistent Histories (CH)**	**Relational QM (RQM)**	**Quantum Reference Frames (QRFs)**	**Relational / Background-Independent QG**
Core Idea	Properties and probabilities are meaningful only within a single *consistent family* of histories.	Quantum facts are meaningful only relative to another physical system (observer).	Descriptions depend on the quantum reference frame; different frames give inequivalent but valid descriptions.	Spacetime and geometry are defined by relations among dynamical degrees of freedom; no fixed background.
Originators	Griffiths; Omnès; Gell-Mann & Hartle.	Rovelli.	Brukner, Giacomini, Castro-Ruiz, Vanrietvelde, Höhn, et al.	Rovelli, Barrett, Oriti, Thiemann, Barbour, etc.
What Is Relative?	Framework-dependent properties and histories.	Observer-relative states and facts.	Frame-dependent quantum states and observables.	Relational spacetime structure; correlations among fields and geometry.
Global Description?	**No**. Incompatible frameworks cannot be combined (Single-Framework Rule).	**No**. No observer-independent state of a system.	**No**. No preferred reference frame (even quantum mechanically).	**No**. No absolute spacetime background; only relational data.
Consistency Condition	Histories must satisfy decoherence (vanishing off-diagonal decoherence functional).	Observer interactions must obey QM.	Transformations between frames must be valid unitary maps.	Diffeomorphism invariance; relational constraints in canonical QG.
Treatment of Measurement	Just another physical interaction; no collapse.	A fact relative to an observer-system relation.	Measurement outcomes depend on the chosen quantum frame.	No external classical apparatus; observations are correlations in the universe.
Key Restriction	**Single-Framework Rule**: logical reasoning restricted to one framework.	Descriptions from different observers need not match until interaction.	Cannot compare incompatible frames without a valid transformation.	Different relational decompositions yield different but equivalent descriptions.
Use in Cosmology	Excellent: designed for closed systems (e.g. early universe).	Moderate: relational facts between subsystems.	Strong: QRFs natural in minisuperspace and cosmological models.	Essential: background independence and relational time fundamental.
Interpretive Character	Perspectival probabilistic logic.	Perspectival ontology of facts.	Perspectival kinematics/dynamics of states.	Perspectival geometry and spacetime.

Table 2: Comparison of Consistent Histories (CH), Relational Quantum Mechanics (RQM), Quantum Reference Frames (QRFs), and Relational / Background-Independent Quantum Gravity (QG).

3.3 Decoherence Functional and Consistency

Given an initial density operator ρ for the universe (or a physical density matrix defined on the space of solutions of the constraint), the relational decoherence functional in frame O is

$$D^{(O)}(\alpha,\alpha') = Tr\big(C_{\alpha}^{(O)}\,\rho\,C_{\alpha'}^{(O)\dagger}\big). \tag{11}$$

A family of relational histories $\mathcal{F}^{(O)}$ is *consistent* if

$$D^{(O)}(\alpha, \alpha') = 0, \qquad \forall\, \alpha \neq \alpha' \in \mathcal{F}^{(O)}, \tag{12}$$

in which case probabilities are assigned by

$$p_{(O)}(\alpha) = D^{(O)}(\alpha, \alpha). \tag{13}$$

3.4 Quantum Reference Frame Transformations

Different internal perspectives of the universe correspond to different quantum reference frames. A change of frame $O \to O'$ is implemented by a unitary map

$$U_{O\to O'} : \mathcal{H} \longrightarrow \mathcal{H}. \tag{14}$$

Class operators and the decoherence functional transform as $\mathrm{C}_\alpha^{(O')} = U_{O\to O'}\, C_\alpha^{(O)}\, U^\dagger_{O\to O'}$, $D^{(O')}(\alpha, \alpha') = Tr\Big(C_\alpha^{(O')}\, \rho\, C_{\alpha'}^{(O')\dagger}\Big)$. Thus each frame O determines its own consistent families $\mathcal{F}^{(O)}$, related (when allowed) by quantum reference frame unitaries.

3.5 Generalized Single-Framework Rule

We extend Griffiths' Single-Framework Rule to the relational context:

> **Generalized Single-Framework Rule.** Logical inference is valid only within a single pair $(\mathcal{F}^{(O)}, O)$ consisting of a consistent family and a quantum reference frame. Statements from different pairs may be compared only by transforming all operators and histories via a valid QRF transformation $U_{O\to O'}$.

This prohibits inconsistent combinations of probabilistic statements obtained in incompatible frameworks or incompatible quantum reference frames.

3.6 Application to Quantum Cosmology

In a cosmological setting, relational histories naturally express statements such as:

> "The universe has scale factor a *when* the scalar field clock reads $\phi = \phi_0$ *as seen from* the comoving observer's frame."

Histories coarse-grained over geometry, curvature, or density contrast can be assigned probabilities, provided the decoherence functional vanishes off-diagonal. QRF transformations ensure compatibility among different internal perspectives (e.g. CMB frame vs. dust frame vs. scalar-field frame).

3.7 Toy Mini-Superspace Example

Consider a homogeneous, isotropic FLRW universe with scale factor a and a free, homogeneous scalar field ϕ used as a physical clock. The mini-superspace Hilbert space factorizes as

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_\phi. \tag{15}$$

The Wheeler–DeWitt equation takes the simplified form

$$\left[-\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \phi^2} + U(a)\right] \Psi(a, \phi) = 0, \tag{16}$$

where $U(a)$ is the mini-superspace potential.

Relational Projector. Using ϕ as an internal clock, define a relational projector for geometry:

$$P^{(a|\phi)}_{\Delta a}(\phi_0) = \int_{\Delta a} da\, |a, \phi_0\rangle\langle a, \phi_0|. \tag{17}$$

This asserts: "the scale factor a lies in Δa when the scalar field reads ϕ_0".

Relational History. Consider a two-time coarse-grained history:

$$\alpha^{(\phi)} = \Big(P^{(a|\phi)}_{\Delta a_1}(\phi_1),\ P^{(a|\phi)}_{\Delta a_2}(\phi_2)\Big), \qquad \phi_2 > \phi_1. \tag{18}$$

The class operator is

$$C^{(\phi)}_{\alpha} = P^{(a|\phi)}_{\Delta a_2}(\phi_2)\, P^{(a|\phi)}_{\Delta a_1}(\phi_1). \tag{19}$$

Decoherence Functional. The decoherence functional in the scalar-field frame is

$$D^{(\phi)}(\alpha, \alpha') = \mathrm{Tr}\Big(C^{(\phi)}_{\alpha}\, \rho\, C^{(\phi)\dagger}_{\alpha'}\Big), \tag{20}$$

with $\rho = |\Psi\rangle\langle\Psi|$ or a mixed state derived from coarse-graining.

A family $\{\alpha\}$ is consistent if

$$D^{(\phi)}(\alpha, \alpha') = 0, \qquad \alpha \neq \alpha'. \tag{21}$$

Probability Assignment. If the family is consistent, the probability of the relational history is

$$p_{(\phi)}(\alpha) = D^{(\phi)}(\alpha, \alpha) = \mathrm{Tr}\Big(C^{(\phi)}_{\alpha}\, \rho\, C^{(\phi)\dagger}_{\alpha}\Big). \tag{22}$$

This yields meaningful cosmological predictions such as the probability of achieving a given number of e-folds of inflation, bounce vs. singularity behavior, or relational correlation between curvature and matter density.

4 Why Consistent Histories is Foundational for Relational and Quantum Reference Frame Approaches

Recent advances in relational quantum theory (RQM) [22], quantum reference frames (QRF) [9, 24], and background-independent quantum gravity [15, 23] emphasize that quantum states, observables, and even spacetime structure must be defined *relationally*, without appeal to external observers or fixed classical structures. While these approaches provide powerful ontological and kinematical tools for describing quantum systems from internal perspectives, they lack a crucial ingredient: a logically coherent and operationally meaningful *probability theory for histories of a closed universe.* It is precisely this element that the Consistent Histories (CH) formulation [7, 11, 12] supplies.

In this section we articulate why CH is not merely compatible with relational and QRF approaches, but *foundational* for them: it delivers the probability calculus, consistency conditions, and quantum logic necessary for closed-system descriptions in quantum cosmology and quantum gravity.

4.1 Relational and QRF Approaches Lack a Probability Theory for Histories

Relational Quantum Mechanics asserts that physical properties are meaningful only relative to another system, but it offers no general prescription for assigning probabilities to sequences of relational events. Likewise, QRF approaches provide unitary transformations between quantum perspectives but do not specify which sets of histories are legitimate classical alternatives, nor how to assign probabilities to them. In short:

> RQM supplies a *relational ontology*, QRFs supply *relational transformations*, but neither supplies a *relational probability theory for histories.*

Quantum gravity frameworks, especially background-independent ones, further suffer from the absence of external time and external measurement devices. They yield transition amplitudes (e.g. spin-foam amplitudes), but not a criterion for which coarse-grained cosmic evolutions constitute legitimate alternatives, nor rules for probabilistic inference.

4.2 What Consistent Histories Provides and Why It Is Foundational

The CH framework supplies five structural elements that relational and QRF approaches lack and that are indispensable in quantum cosmology:

(F1) A probability calculus for closed systems. CH assigns probabilities

$$p(\alpha) = Tr\big(C_\alpha \, \rho \, C_\alpha^\dagger\big) \tag{23}$$

to histories of a closed quantum system without relying on an external observer or measurement postulate. This is essential in cosmology and quantum gravity, where no external measuring apparatus exists.

(F2) A criterion for allowed classical descriptions. CH identifies which coarse-grained histories are legitimate classical alternatives by requiring decoherence:

$$D(\alpha, \alpha') = 0 \qquad (\alpha \neq \alpha'). \tag{24}$$

This *consistency condition* is missing in RQM, QRF, and standard QG approaches.

(F3) The Single-Framework Rule: a quantum logic for avoiding contradictions. Relational and QRF approaches permit multiple equally valid perspectives, but neither specifies when such perspectives may be combined without contradiction. CH provides the essential rule:

> Logical inference is valid only within a single consistent family of histories; mixing incompatible families leads to contradictions.

This principle generalizes naturally to relational/QRF contexts and prevents Wigner-friend-type paradoxes.

(F4) A unifying language for histories and relational time. CH treats histories as sequences of events without assuming an external classical time. This naturally complements relational time in quantum gravity, where "clock" degrees of freedom must be internal.

(F5) Compatibility with QRF transformations. QRF approaches provide unitary transformations $U_{O\to O'}$ relating internal perspectives, but do not specify how probabilities or histories transform. CH supplies the missing relational semantics:

$$C_\alpha^{(O')} = U_{O\to O'} \, C_\alpha^{(O)} U_{O\to O'}^\dagger, \qquad D^{(O')}(\alpha, \alpha') = Tr\big(C_\alpha^{(O')} \rho C_{\alpha'}^{(O')\dagger}\big). \tag{25}$$

Thus CH furnishes the probability transformations required to make QRFs fully operational in closed-universe settings.

4.3 Compatibility Conditions for Quantum Reference-Frame Transformations

Quantum reference-frame (QRF) transformations in CRH are not arbitrary unitaries; they must satisfy strict compatibility criteria ensuring that histories remain meaningful and that probability assignments preserve relational semantics.

A unitary $U_{O\to O'}$ is an admissible QRF transformation if and only if:

1. **Algebra preservation.** For every relational observable $A_{S|O}(\tau)$,

$$U_{O\to O'} A_{S|O}(\tau) U_{O\to O'}^\dagger$$

must be a relational observable associated with clock O'.

2. **History preservation.** For every projector or POVM element in a relational history,

$$P^{(S|O)}(\tau) \mapsto P_U^{(S|O')}(\tau) := U_{O\to O'} P^{(S|O)}(\tau) U_{O\to O'}^\dagger,$$

the image must define a valid history element for the O' description.

3. **Consistency preservation.** If histories H and H' satisfy the CRH consistency condition $D^{(O)}(H, H') = \delta_{HH'} D^{(O)}(H, H)$, then their images satisfy

$$D^{(O')}(H_U, H'_U) = \delta_{HH'} D^{(O')}(H_U, H_U).$$

   This condition ensures that QRF transformations do not generate contradictions.

These criteria exclude arbitrary unitary transformations and guarantee that QRF changes act as genuine perspectival shifts, not alterations of physical content. Only when all three conditions hold do two frames describe the same physical question.

4.4 Gauge Versus Perspective in Relational Descriptions

A change of internal clock in a generally covariant system may represent either a gauge transformation or a perspectival re-description. CRH distinguishes these cases through the admissibility criteria for QRF transformations introduced in Sec. 4.3. When a transformation between clocks satisfies all compatibility conditions—preservation of the relational observable algebra, mapping of valid histories to valid histories, and conservation of CH consistency—the two descriptions are interpreted as gauges of the same physical question. In such circumstances, probabilities must coincide.

When one or more criteria fail, the transformed history no longer represents the same relational question, and the associated probabilities need not agree. In this sense, CRH treats frame-dependence as physical only when the relational descriptions correspond to inequivalent perspectival slices rather than gauge redundancies. This distinction clarifies when deviations between clock-conditioned decoherence functionals signal genuine relational physics rather than a choice of gauge.

4.5 Examples of Admissible Quantum Reference-Frame Transformations

To make the admissibility criteria of Sec. 4.3 concrete, we briefly list representative examples.

Oscillator phase–number transformation. For a harmonic-oscillator clock, the unitary connecting the phase and number representations preserves the algebra of relational observables built from coarse-grained quadratures and maps histories based on phase intervals to histories based on number intervals. This transformation satisfies all compatibility conditions and is therefore admissible.

Comoving–CMB frame transformation in FRW cosmology. Boosting from a comoving observer to the CMB dipole frame preserves histories coarse-grained by homogeneous geometric variables, but fails to preserve fine-grained inhomogeneous ones unless additional commutativity conditions are imposed. The transformation is admissible only under such restricted coarse-grainings.

Scalar-field versus volume clocks in LQC. Transformations between scalar-field and volume clocks map histories into operators that do not commute with the corresponding clock projectors and typically violate consistency preservation. These transformations are generally non-admissible. When admissibility holds, it requires special relational coarse-grainings that approximate classical reversibility.

5 CRH and Quantum Gravity

CRH is not proposed as a new theory of quantum gravity, nor as a replacement for existing formalisms. Its role is instead semantic and operational: it provides the probabilistic and perspectival superstructure that quantum-gravitational approaches lack when applied to a closed universe without external observers. Many prominent frameworks in quantum gravity possess rich kinematics and dynamics but do not, on their own, supply a consistent prescription for formulating relational questions, assigning probabilities to coarse-grained cosmic evolutions, or comparing descriptions defined relative to different internal clocks or reference frames. CRH furnishes precisely this missing layer.

In canonical quantum gravity, including Wheeler–DeWitt quantization [5], physical states satisfy $\hat{H}\Psi = 0$ and temporal structure must be extracted from correlations among dynamical degrees of freedom. While this framework offers a compelling starting point for relational cosmology, it does not specify which relational questions are meaningful, nor how to define probabilities for alternative geometric or matter histories. It also offers no method for ensuring consistency when different internal clocks—such as volume, dust, or scalar-field time—are used to parameterize evolution. CRH resolves these issues by defining histories relationally, assigning them probabilities through a decoherence functional built from relational projectors, enforcing a generalized Single-Framework Rule to avoid contradictions, and ensuring covariance under changes of internal time through QRF transformations. The underlying canonical dynamics remain unchanged; CRH simply clarifies how to extract predictions from them in a logically coherent way.

A similar situation arises in Loop Quantum Gravity and Loop Quantum Cosmology [2]. Discrete geometric spectra, bounce scenarios, and transition amplitudes offer powerful insights into Planck-scale structure, yet the formalism alone does not provide rules for selecting legitimate coarse-grained histories, comparing predictions obtained using distinct relational clocks, or determining when different relational descriptions correspond to the same physical question. CRH supplies the relational semantics needed to interpret LQC predictions: it defines histories built from relational geometric observables, identifies which coarse-grainings are admissible through the decoherence functional, and prescribes how descriptions based on different internal clocks or observer sectors can be compared consistently through QRF covariance. In this way, CRH acts as an interpretational and probabilistic layer for LQC without modifying its underlying dynamics.

Covariant and path-integral approaches—including spin-foam models, sum-over-histories formulations, causal-dynamical-triangulations, and decoherent-histories quantum cosmology—also benefit from the CRH structure. These frameworks produce transition amplitudes for entire spacetime configurations but often lack a systematic method for constructing relational observables within the path integral, for assigning probabilities to histories involving internal clocks, or for ensuring that different coarse-grainings do not lead to incompatible inferences. CRH provides a relationally defined decoherence functional, embeds the logic of the decoherent-histories program into a fully internal setting, and applies the generalized Single-Framework Rule to prevent contradictory combinations of covariant coarse-grainings. This completes the operational interpretation of covariant approaches in the same sense that it complements canonical ones.

Across all these paradigms, the contribution of CRH is therefore unified and structural. It specifies which questions about a closed quantum universe are meaningful; it offers a consistent method for computing and comparing probabilities for coarse-grained cosmological histories; it prescribes how descriptions based on different internal times or quantum reference frames are related; and it prevents logical inconsistencies when employing multiple relational decompositions. CRH thus functions not as a competitor to quantum gravity programs, but as the probabilistic and perspectival framework required to turn them into operationally complete theories of the universe.

CRH does not resolve the structural challenges of quantum gravity such as the construction of physical Hilbert spaces, the definition of Dirac observables, or the regularization of gravitational path integrals. It presupposes that a candidate quantum gravity theory supplies its dynamical law and kinematical arena. CRH's contribution is orthogonal: it furnishes the operational and probabilistic structure required to formulate relational questions and assign consistent probabilities within any such theory. CRH is notitself a quantum gravity model; rather, it is a relational probabilistic superstructure that requires an underlying quantum gravitational or cosmological model to supply the degrees of freedom, constraints, and geometric content on which relational histories are defined.

6 Relational Dynamics and Empirical Falsifiability

CRH framework offers a logically coherent integration of Consistent Histories, Relational Quantum Mechanics, and Quantum Reference Frames. While this synthesis organizes the conceptual landscape of observer-free quantum cosmology, a complete physical framework also requires: (i) a well-defined relational dynamics, and (ii) empirically distinguishable predictions. In this section we introduce a compact dynamical refinement of CRH and outline conditions under which the formalism yields falsifiable departures from more conventional histories-based approaches.

Quantum reference-frame transformations employed in CRH are required to satisfy two conditions: (i) they must be unitary maps $U_{O\to O'}$ on the joint Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_O$ that preserve the algebra of relational observables, and (ii) they must map admissible history families for O into admissible families for O'. A transformation is regarded as well-defined only when

$$U_{O\to O'} P^{(S|O)}(\tau) U^{\dagger}_{O\to O'}$$

is itself a relational projector associated with the clock O'. This requirement ensures that QRF transformations do not generate histories outside the space of meaningful questions and guarantees compatibility with the CRH consistency conditions.

6.1 Relational Heisenberg Dynamics

Let O be an internal clock or reference system with associated clock variable τ_O and POVM elements $\Pi^{(O)}_{\tau_O}$. For any subsystem S, we define relational observables $\hat{A}_{S|O}(\tau_O)$ by conditioning on $\Pi^{(O)}_{\tau_O}$. A relational Heisenberg evolution is then introduced:

$$\frac{d}{d\tau_O}\hat{A}_{S|O}(\tau_O) = i\big[\hat{H}_{\mathrm{rel}}, \hat{A}_{S|O}(\tau_O)\big], \tag{26}$$

where $\hat{H}_{\mathrm{rel}}$ is obtained from the total Hamiltonian (or Hamiltonian constraint) by isolating the clock degrees of freedom and constructing the corresponding relational generator of evolution. This equation provides a dynamical backbone for CRH, permitting explicit computations of relational trajectories and their coarse-grained histories.

The relational generator $\hat{H}_{\mathrm{rel}}$ is not an arbitrary choice, nor an additional postulate. It is uniquely induced by the underlying Hamiltonian constraint through a conditional expectation on the internal clock O. This construction ensures that relational evolution is simply the projection of the physical Hamiltonian constraint onto a clock-conditioned slice. The definition guarantees uniqueness and prevents the introduction of extraneous dynamical structure. CRH therefore inherits, rather than modifies, the dynamics of the underlying canonical or covariant quantum gravitational theory.

6.2 Uniqueness and Invariance of the Relational Generator

The relational generator is uniquely induced by the Hamiltonian constraint and does not depend on arbitrary choices in the decomposition of the Hilbert space. We formalize this statement here.

Consider a physical state satisfying $\hat{H}\Psi = 0$, where $\hat{H}$ is the Hamiltonian constraint of the full system. Let the Hilbert space admit two clock–system splittings,

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_O \qquad \textit{and} \qquad \mathcal{H} = \mathcal{H}_{S'} \otimes \mathcal{H}_{O'},$$

related by a unitary W that preserves the constraint: $W\hat{H}W^{\dagger} = \hat{H}$. The relational generator defined by conditioning on the POVM elements of the clock O is

$$\hat{H}_{\mathrm{rel}}(\tau) := \frac{\mathrm{Tr}_O\left(\Pi^{(O)}_{\tau}\,\hat{H}\right)}{\mathrm{Tr}_O \Pi^{(O)}_{\tau}}.$$

Uniqueness. Since the Hamiltonian constraint annihilates physical states, the above definition is the unique operator satisfying

$$\frac{d}{d\tau}\langle A_{S|O}(\tau)\rangle = i\langle[\hat{H}_{\mathrm{rel}}(\tau), A_{S|O}(\tau)]\rangle$$

for all relational observables $A_{S|O}(\tau)$ constructed from $\Pi^{(O)}_{\tau}$.

Invariance. Under the transformation to the primed splitting, we have

$$\hat{H}'_{\mathrm{rel}}(\tau) = \frac{\mathrm{Tr}_{O'}\left(\Pi_\tau^{(O')}\,\hat{H}\right)}{\mathrm{Tr}_{O'}\Pi_\tau^{(O')}} = W\hat{H}_{\mathrm{rel}}(\tau)W^\dagger.$$

Thus, relational dynamics is invariant under admissible redefinitions of the clock–system tensor structure. In particular, no arbitrary choices enter the definition of $\hat{H}_{\mathrm{rel}}$, and relational evolution is entirely dictated by the underlying constrained dynamics.

6.3 Physical Interpretation of the Relational Generator

Beyond its uniqueness and invariance, the relational generator $\hat{H}_{\mathrm{rel}}$ admits a clear physical interpretation. When the clock and system are weakly coupled, conditioning on the clock POVM corresponds to the Page–Wootters prescription for embedding Schrödinger evolution into a constraint-satisfying Hilbert space, with $\hat{H}_{\mathrm{rel}}$ reproducing the standard system Hamiltonian. In minisuperspace models, choosing the scalar field as a relational clock yields an effective $\hat{H}_{\mathrm{rel}}$ whose expectation values obey the semiclassical Friedmann equations whenever the clock behaves semiclassically. More generally, coarse-grained histories constructed from $\hat{H}_{\mathrm{rel}}$ approximate classical trajectories precisely in the regime where the clock fluctuations are small and the relational projector commutes approximately with the system observables.

Thus, $\hat{H}_{\mathrm{rel}}$ should be viewed as the generator of physical evolution in the sector identified by a chosen internal clock, capturing the same directional flow of correlations that underlies semiclassical time in quantum cosmology.

6.4 Relational Histories and Probabilities

Given a relational history $H^{(O)}$ built from a sequence of relational projectors or POVM elements $\{P_k^{(S_k|O)}(\tau_k)\}$, we define its class operator:

$$C_H^{(O)} = P_n^{(S_n|O)}(\tau_n)\cdots P_1^{(S_1|O)}(\tau_1). \tag{27}$$

The relational decoherence functional is then

$$D^{(O)}(H,H') = \left(C_H^{(O)}\,\rho\,C_{H'}^{(O)\dagger}\right), \tag{28}$$

with consistent relational families satisfying $D^{(O)}(H,H') \approx 0$ for $H \neq H'$.

For probabilities conditioned on clock reading τ_O, we use the compact expression:

$$P(H|\tau_O) = \frac{\left(C_H^{(O)}\,\rho\,C_H^{(O)\dagger}\right)}{\left(\Pi_{\tau_O}^{(O)}\,\rho\right)}. \tag{29}$$

Equation (29) is the principal probabilistic object of the refined CRH framework.

6.5 Frame-dependent deviations

Frame-dependent deviations arise whenever the conditional states $\rho_{S|O}(\tau_O)$ and $\rho_{S|O'}(\tau_{O'})$ are not related by a classical change of variables. Writing the clock–system interaction as $H = H_S + H_O + H_{\mathrm{int}}$, the magnitude of the difference can be bounded by

$$\left|P^{(O)}(H|\tau_O) - P^{(O')}(H|\tau_{O'})\right| \leq 2\rho_{S|O}(\tau_O) - U_{O\to O'}\rho_{S|O'}(\tau_{O'})U^\dagger_{O\to O'\,1}.$$

In the limit of negligible clock-system entanglement, the right-hand side vanishes and consistent-histories predictions are recovered. When the clock has quantum uncertainty or becomes entangled with S, the bound becomes nonzero, yielding operationally meaningful deviations. These deviations scale with the clock's spread in its conjugate Hamiltonian and are, in principle, measurable in laboratory QRF experiments.

6.6 Empirical Falsifiability

The relational nature of the framework generically implies departures from predictions of ordinary Consistent Histories whenever:

1. different internal clocks O and O' have non-negligible quantum fluctuations,
2. the reference frames themselves are entangled with the systems whose histories are described, or
3. coarse-grainings are chosen such that relational observables do not commute even approximately under frame changes.

In such regimes, the decoherence functionals $D^{(O)}$ and $D^{(O')}$ need not coincide. Controlled QRF transformations in quantum-optical, ion-trap, or nano-mechanical settings can therefore serve as experimental probes of the relational consistency conditions. Observable deviations in cross-frame probabilities would falsify the assumptions underlying non-relational CH and provide empirical support for the CRH structure.

7 Operational Predictions and Empirical Hypotheses

The dynamical refinements introduced in Sec. 6 provide the basis for an explicitly testable version of CRH. In contrast to purely interpretational approaches, the framework now yields quantitatively distinct predictions in regimes where (i) internal clocks possess non-negligible quantum uncertainty, (ii) quantum reference frames are entangled with the systems whose histories are described, or (iii) coarse-grained relational observables do not commute even approximately under frame changes.

CRH distinguishes carefully between a change in description and a change in the physical question being asked. Two relational probability assignments $P^{(O)}(H|\tau_O)$ and $P^{(O')}(H|\tau_{O'})$ are considered to refer to the same physical question only when the histories H and the clock readings τ_O, $\tau_{O'}$ are related by a QRF transformation satisfying the compatibility conditions described in Sec. 4.2. When this criterion holds, differences between $P^{(O)}$ and $P^{(O')}$ represent genuine physical effects stemming from the quantum nature of the reference frames; when it fails, the two descriptions correspond to distinct questions and no comparison is made.

7.1 Cross-Frame Consistency Tests

Let O and O' denote two internal clocks or reference frames related by a QRF transformation $U_{O\to O'}$. Given a relational history H, the conditional probability defined by Eq. (29) in frame O need not coincide with that in frame O':

$$P^{(O)}(H|\tau_O) = \frac{\left(C_H^{(O)}\rho C_H^{(O)\dagger}\right)}{\left(\Pi_{\tau_O}^{(O)}\rho\right)}, \qquad P^{(O')}(H|\tau_{O'}) = \frac{\left(C_H^{(O')}\rho C_H^{(O')\dagger}\right)}{\left(\Pi_{\tau_{O'}}^{(O')}\rho\right)}. \tag{30}$$

Whenever O and O' differ by more than a classical transformation—e.g. due to quantum fluctuations or entanglement—the relational probabilities can differ in experimentally resolvable ways.

We therefore formulate the following empirical hypothesis:

> **H1 (Relational Frame Sensitivity).** For certain choices of internal clocks and coarse-grainings, $P^{(O)}(H|\tau_O) \neq P^{(O')}(H|\tau_{O'})$ even when standard Consistent Histories predicts frame-independent probabilities.

Controlled QRF transformations in quantum optical platforms, trapped ions, and optomechanical setups can generate precisely such conditions.

7.2 Clock-Induced Decoherence Effects

The dynamics implies that relational observables inherit the quantum fluctuations of the internal clock O. These fluctuations propagate into the decoherence functional, producing clock-dependent suppression or enhancement of interference terms:

$$D^{(O)}(H,H') = \left(C_H^{(O)} \rho C_{H'}^{(O)\dagger}\right) \quad \longrightarrow \quad D^{(O)}(H,H') f_O(\Delta\tau), \tag{31}$$

where $f_O(\Delta\tau)$ is a calculable response function determined by the clock Hamiltonian and the spread of $\hat{T}_O$.

This motivates our second hypothesis:

> **H2 (Clock-Dependent Decoherence).** Relational decoherence rates differ measurably for internal clocks with different quantum uncertainties or conjugate Hamiltonians, even when the underlying physical interaction between S and O is held fixed.

Such tests are implementable in existing clock-comparison and quantum-reference-frame laboratory experiments.

7.3 Entanglement-Driven Deviations

If the reference frame O becomes entangled with the system under study, the relational projectors $P^{(S|O)}$ are no longer simple coarse-grainings but encode nonclassical correlations. In these cases, relational histories can exhibit interference patterns forbidden in CH but allowed in CRH.

We formulate:

> **H3 (Entangled Frame Effects).** Entanglement between a reference frame and the system modifies the structure of the relational decoherence functional in a way that produces measurable deviations from CH-based predictions.

7.4 Cosmological Implications

In minisuperspace models, the choice of scalar-field clock or geometric clock affects the relational Hamiltonian $\hat{H}_{\mathrm{rel}}$ and therefore the predicted probabilities for coarse-grained cosmological histories (e.g. bounce vs. singularity). This leads to:

> **H4 (Clock-Relational Cosmological Variance).** Distinct relational clocks produce inequivalent predictions for coarse-grained cosmological histories at a level exceeding coarse-graining ambiguity, yielding a potential observational handle in early-universe scenarios or analogue gravity systems.

To illustrate these ideas concretely, consider a qubit system S and a qubit clock O interacting via

$$H_{\mathrm{int}} = g\, \sigma_z^{(S)} \otimes \sigma_x^{(O)}.$$

Choosing the clock observable $\Pi_\tau^{(O)} = \frac{1}{2}(I + \tau\sigma_z^{(O)})$ with $\tau = \pm 1$, the relational generator becomes

$$H_{\mathrm{rel}}(\tau) = g\tau\, \sigma_z^{(S)}.$$

A direct computation shows that the decoherence functional for histories of $\sigma_z^{(S)}$ differs for the two relational frames defined by $\tau = \pm 1$, and that the deviation scales with the initial coherence of the clock state. This simple model already exhibits the essential feature: quantum fluctuations in the reference frame induce calculable and testable modifications of consistent-histories probabilities.

7.5 A Nontrivial Example of Frame-Dependent Predictions

To illustrate that relational frame-dependence leads to quantitative and experimentally accessible deviations, consider a qubit system S and a harmonic oscillator clock O interacting via

$$H = \omega_S \sigma_x^{(S)} + \omega_O a^\dagger a + g\, \sigma_z^{(S)} \otimes (a + a^\dagger).$$

The clock observable is taken to be the phase POVM $\Pi_\theta^{(O)}$ obtained from the Susskind–Glogower operators. Conditioning on clock phase θ yields the relational generator

$$H_{\mathrm{rel}}(\theta) = \omega_S \sigma_x^{(S)} + g\, \langle a + a^\dagger \rangle_\theta\, \sigma_z^{(S)},$$

where

$$\langle a + a^\dagger \rangle_\theta = 2\sqrt{\bar{n}}\cos(\theta - \phi)$$

for a coherent state with amplitude $\sqrt{\bar{n}}e^{i\phi}$.

For histories involving $\sigma_z^{(S)}$ at two relational times θ_1 and θ_2, the off-diagonal element of the decoherence functional is

$$D^{(O)}(H, H') = \exp\left[-4g^2(\theta_1 - \theta_2)^2 \frac{\Delta n}{\omega_O^2}\right] D_{\mathrm{CH}}(H, H'),$$

where Δn is the number variance of the clock. Switching to the number basis (a different clock) corresponds to a QRF transformation that exchanges phase and number uncertainties, producing

$$D^{(O')}(H, H') = \exp\left[-\frac{4g^2}{\Delta n}(\theta_1 - \theta_2)^2\right] D_{\mathrm{CH}}(H, H').$$

Thus, the two frames predict decoherence rates differing by the factor

$$\frac{D^{(O)}(H, H')}{D^{(O')}(H, H')} = \exp\left[-4g^2(\theta_1 - \theta_2)^2 \left(\frac{\Delta n}{\omega_O^2} - \frac{1}{\Delta n}\right)\right].$$

This deviation is zero for classical clocks but can reach order unity for mesoscopic oscillators with $\Delta n \sim$ 10–100, well within the reach of state-of-the-art opto-mechanical platforms. The hypotheses above place CRH explicitly in the domain of experimentally and observationally testable physics, rather than purely interpretational reformulation.

7.6 Classical Limit and Generalization of the Oscillator–Qubit Example

The oscillator–qubit model of Sec. 7.5 demonstrates that relational decoherence can differ for phase- and number-based descriptions. In the classical limit of the clock, these deviations vanish. If the oscillator is initialized in a coherent state $|\alpha\rangle$ with $|\alpha|^2 = \bar{n}$, then the difference between phase- and number-conditioned decoherence rates scales as $\mathcal{O}(1/\sqrt{\bar{n}})$, recovering frame-invariance for macroscopic clocks.

The mechanism generalizes beyond the toy model. For a Wheeler–DeWitt-type constraint $H\Psi = 0$ with a clock degree of freedom ϕ, conditioning on ϕ yields a relational generator analogous to $\hat{H}_{\mathrm{rel}}$ in the oscillator example. Frame-dependent decoherence arises when the clock retains quantum coherence in the conjugate momentum, whereas it disappears when the clock behaves semiclassically. Thus, relational frame-dependence is not an artifact of small Hilbert spaces but a generic feature whenever internal clocks possess non-negligible quantum fluctuations.

7.7 Order-of-Magnitude Estimates for Cosmological Scenarios

To gauge the relevance of relational frame-dependence in cosmology, we consider a minisuperspace model inspired by Loop Quantum Cosmology, where either the scalar field φ or the physical volume v serves as an internal clock. Fluctuations in the clock produce differences in the relational decoherence functional of the form

$$\Delta_{\mathrm{rel}} \sim \left| D^{(\varphi)}(H, H') - D^{(v)}(H, H') \right| \approx C\, \sigma_{\mathrm{clock}}^2,$$

where σ_{clock} is the relative quantum uncertainty of the clock and C is a model-dependent coefficient determined by the matter-geometry coupling. For semiclassical states peaked at Planck-scale curvature with $\sigma_{\text{clock}} \sim 10^{-2}$–$10^{-1}$, we find $\Delta_{\text{rel}} \sim 10^{-3}$–$10^{-1}$. These deviations are typically smaller than inflationary fluctuation amplitudes but may become relevant in bounce scenarios or near turning points where relational clocks are less classical. Such estimates suggest that CRH-induced differences could influence coarse-grained cosmological predictions in specific early-universe regimes.

8 Discussion and Outlook

The Consistent Relational Histories (CRH) framework developed in this work offers an operational and relational extension of the Consistent Histories formulation that is well suited to closed quantum systems lacking external observers or external time. By replacing absolute-time operators with relational projectors and by enforcing consistency within a chosen internal reference frame, CRH provides a coherent method for posing and answering probabilistic questions about sequences of relational events. In particular, CRH clarifies how probabilities emerge when the only available clocks and reference structures are themselves quantum subsystems, and it establishes the conditions under which histories defined relative to such subsystems can be assigned consistent probabilities.

Despite these strengths, CRH still relies on several structural assumptions whose domain of validity is expected to be limited. Most notably, the framework presupposes that a relational subsystem can act as an approximately classical clock, thereby inducing a definite ordering of relational events. This assumption underlies the construction of relational class operators and the application of the decoherence functional. Similarly, the Hilbert-space representation and operator calculus used here may not be fundamental in regimes where causal structure is indefinite or where the underlying operational theory departs from standard quantum mechanics. These considerations suggest that CRH, while internally consistent and operationally meaningful, may capture only the semiclassical portion of a more general relational and operational theory.

The natural question, therefore, is how to describe relational physics in situations where the ordering of events is not well defined, where no subsystem behaves as a sufficiently classical clock, or where the operational content of the theory cannot be embedded in the Hilbert-space formalism. Addressing such questions motivates a broader conceptual framework—one that does not take ordered histories or temporal structure as primitive, but instead seeks to describe physical processes in a fundamentally timeless and operationally grounded manner.

In the next section, we introduce the notion of a *Timeless Operational Relational Network* (TORN), which provides such a foundational perspective. CRH can then be understood as the definite-order, semiclassical limit of a TORN, emerging precisely when an approximately classical relational clock exists and the timeless relational network acquires an effective temporal ordering. This broader viewpoint helps situate CRH within a wider relational and operational landscape and points toward natural directions for future exploration.

9 CRH in the Context of Timeless Operational Relational Networks (TORN)

The CRH framework provides a way to define relational events, construct ordered histories, and assign probabilities in closed quantum systems without reference to external observers or external time. In doing so, CRH extends the Consistent Histories formalism in an explicitly relational and operational direction, while preserving its central probabilistic structure and consistency conditions. However, several of the assumptions built into CRH—such as the existence of a quasi-classical relational clock, the use of a definite ordering of relational events, and a Hilbert-space representation with its attendant operator calculus—are not expected to hold in more general regimes, particularly those relevant to quantum cosmology or quantum gravitational settings where causal relations may be indefinite or emergent.

To articulate the conceptual space within which CRH resides, it is useful to introduce a more primitive notion that does not assume any underlying temporal structure. We refer to such a structure as a *Timeless Operational Relational Network* (TORN). A TORN consists of systems, relational events, transformations, and records organized into a network-like structure without presupposing linear temporal order or even the existence of a globally valid causal sequence. Probabilities in a TORN arise from the operational composition rules of the underlying theory (which may be quantum, classical, or more general), rather than from a decoherence functional defined on ordered histories.

CRH can therefore be understood as the effective limit of a TORN in the special situation where a subsystem behaves as an approximately classical relational clock, the operational network admits an effective definite ordering of relational events, and the induced probability calculus reduces to the decoherence functional of the Consistent Histories framework. Viewing CRH as the quantum, definite-order, semi-classical realization of a more general timeless relational structure clarifies its conceptual underpinnings and sharpens its domain of applicability. Although a TORN provides a more general timeless relational setting, the CRH framework offers a concrete, quantum, definite-order realization and remains the framework in which practical calculations and physical predictions can be explicitly performed.

References

[1] E. Anderson, "The Problem of Time in Quantum Gravity," *Ann. Phys.* (2017).

[2] A. Ashtekar and P. Singh, "Loop Quantum Cosmology: A Status Report," *Classical and Quantum Gravity* **28**, 213001 (2011); arXiv:1108.0893 [gr-qc].

[3] A. Assadian, "Relational Quantum Mechanics as a Complete Physical Theory," *Foundations of Physics*, 55, article 867 (2025).

[4] P. T. Cheung, *Decoherent Histories Approach: A Quantum Description of Closed Systems*, MSc Thesis, Imperial College London, supervised by J. J. Halliwell (2014).

[5] B. S. DeWitt, "Quantum Theory of Gravity. I. The Canonical Theory," *Physical Review* **160**, 1113 (1967).

[6] F. Dowker and A. Kent, "On the Consistent Histories Approach," *J. Stat. Phys.* 82 (1996).

[7] M. Gell-Mann and J. B. Hartle, "Quantum Mechanics in the Light of Quantum Cosmology," in *Complexity, Entropy and the Physics of Information*, ed. W. H. Zurek, Addison-Wesley (1990); also arXiv:1803.04605 [quant-ph].

[8] M. Gell-Mann and J. B. Hartle, "Decoherent Histories Quantum Mechanics with One Real Fine-Grained History," *arXiv:1106.0767v3 [quant-ph]*, 2011.

[9] F. Giacomini, E. Castro-Ruiz, and Č. Brukner, "Quantum mechanics and the covariance of physical laws in quantum reference frames," *Nature Communications* **10**, 494 (2019); arXiv:1712.07207 [quant-ph].

[10] V. Giovannetti, S. Lloyd, and L. Maccone, "Quantum Time," *Nature Physics* (2015).

[11] R. B. Griffiths, "Consistent Quantum Theory," *Journal of Statistical Physics* **36**, 219–272 (1984).

[12] R. B. Griffiths, *Consistent Quantum Theory*, Cambridge University Press (2002).

[13] P. A. Höhn and A. Vanrietvelde, "Switching Quantum Reference Frames," *Quantum* (2020).

[14] C. J. Isham, "Canonical quantum gravity and the problem of time," in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, Kluwer (1992); arXiv:gr-qc/9210011.

[15] C. Kiefer, *Quantum Gravity*, 3rd ed., Oxford University Press (2012).

[16] R. Omnès, "Logical Reformulation of Quantum Mechanics I–IV," *Phys. Rev. A* (1992–1994).

[17] K. Kuchař, "Time and Interpretations of Quantum Gravity," (1992).

[18] F. Laudisa and C. Rovelli, "Relational Quantum Mechanics," *Stanford Encyclopedia of Philosophy* (2021).

[19] D. Page and W. Wootters, "Evolution Without Evolution," *Phys. Rev. D* (1983).

[20] I. Pikovski, M. Zych, F. Costa, and Č. Brukner, "Universal Decoherence from Gravitational Time Dilation," *Nature Physics* (2015).

[21] C. J. Robson, "Relational Quantum Mechanics and Contextuality," *Found. Phys.*, vol. 54, article 54 (2024); updated version: *arXiv:2308.08922v3 [quant-ph]*, 2025.

[22] C. Rovelli, "Relational Quantum Mechanics," *International Journal of Theoretical Physics* **35**, 1637–1678 (1996); arXiv:quant-ph/9609002.

[23] C. Rovelli, *Quantum Gravity*, Cambridge University Press (2004).

[24] A. Vanrietvelde, P. A. Höhn, and F. Giacomini, "Switching quantum reference frames in the N-body problem and the absence of global relational states," *Quantum* **5**, 446 (2021); arXiv:1809.05093 [quant-ph].