The Sound of Creation

Lecture 1: Waves We Know

From Guitar Strings to the Speed of Sound

Tayur Lectures on Physics (Necklace 2)

Contents

1 Why Waves?

Before we can appreciate the extraordinary story of sound shaping the cosmos, we need to speak the language of waves fluently. This first lecture is deliberately earthbound. We will study the vibration of a guitar string, the resonance of a pipe organ, and the propagation of pressure pulses through air. Along the way we will derive the wave equation from first principles, extract the speed of sound from the bulk modulus and density of a medium, carefully distinguish phase velocity from group velocity, and examine how energy is carried by waves. Every concept introduced here will reappear, transformed but recognisable, when we reach the photon–baryon plasma of the early universe.

The history of wave physics is a story worth pausing over. It begins, as so much of classical mechanics does, with Galileo and the pendulum in the late sixteenth century. Galileo, confined to house arrest after his confrontation with the Inquisition, reportedly spent years exploring the relationship between the length of a pendulum and its period of oscillation. This was a result equivalent to saying that a restoring force proportional to displacement produces simple harmonic motion—the primitive seed of all wave physics. The next major step came with Marin Mersenne, a French friar and polymath, whose 1636 treatise *Harmonie Universelle* reported systematic measurements of vibrating strings. Mersenne's laws established by experiment that the frequency of a stretched string is proportional to $\sqrt{T}$, inversely proportional to L, and inversely proportional to $\sqrt{\mu}$. Remarkably, Mersenne was also among the first to measure the actual speed of sound, obtaining roughly $450\,\mathrm{m\,s^{-1}}$—an overestimate, but the correct order of magnitude.

Isaac Newton, in Book II of the *Principia Mathematica* (1687), attempted the first theoretical derivation of the speed of sound in air and obtained $v \approx 290\ \mathrm{m\,s^{-1}}$. The measured value at his time was close to $340\,\mathrm{m\,s^{-1}}$. Newton was aware of the discrepancy but rather than questioning his assumptions he added *ad hoc* corrections invoking the finite size of air molecules and vapour content. The true reason remained elusive for over a century. It fell to Pierre-Simon Laplace, in a brief 1816 note to the *Annales de Chimie et de Physique*, to identify the error. Newton had assumed that the compressions in a sound wave were *isothermal*: as a parcel of air is squeezed, heat flows away fast enough to keep its temperature constant. Laplace pointed out that the timescale of a sound wave (milliseconds) is far too short for significant heat exchange between adjacent parcels. The compressions are therefore *adiabatic*: no heat is exchanged, and the temperature rises during compression, making the gas locally stiffer than the isothermal approximation suggests. Replacing Newton's isothermal bulk modulus $B = P$ with the adiabatic $B = \gamma P$ (where $\gamma \approx 1.4$ for diatomic air) immediately yields $v_{\text{correct}} = \sqrt{\gamma}\, v_{\text{Newton}} \approx 1.183 \times 290 \approx 343\ \mathrm{m\,s^{-1}}$, in excellent agreement with measurement. That single insight—that the relevant thermodynamic process matters—is a theme that will echo through this entire course. In the cosmic plasma, the relevant compression is neither isothermal nor simply adiabatic, but is governed by the tight radiative coupling between photons and baryons; getting the effective bulk modulus right is everything.

2 What Is a Wave? Pattern Versus Particle

Before we write a single equation, we need to be clear about something that confuses almost everyone at first encounter: *a wave is a travelling pattern, not a travelling substance.* This distinction is so important—and so easy to blur—that it deserves its own section.

2.1 The Stadium Wave: An Everyday Illustration

Picture a sold-out football stadium. At some point a fan in one section stands up, raises their arms, and sits back down. The person next to them follows a moment later, then the next, and so on around the arena. What you see from the air is a ripple of raised arms sweeping around the stadium at impressive speed. This is the "Mexican wave," and it is one of the cleanest illustrations of what a wave actually is.

Notice what is *not* happening: none of the fans are moving around the stadium. Each person is simply standing up and sitting back down—oscillating vertically in their own seat, returning to exactly where they started after the wave has passed through them. The pattern of raised arms travels around the stadium; the fans do not. The wave is a coordinated sequence of local motions, not the transport of any person or seat from one place to another.

This is the essence of all wave motion. In every wave, two things coexist: the **medium**—the fans, the string, the air, the plasma—which oscillates locally and returns to its resting state; and the **wave pattern**—the disturbance shape—which propagates through the medium, potentially at very high speed. These two things can travel at radically different speeds, and separating them conceptually is the first step toward understanding waves quantitatively.

2.2 Transverse Waves: The Displacement Is Perpendicular

A guitar string provides the next illustration. When you pluck a string, each tiny segment of string moves *up and down* (or side to side), perpendicular to the length of the string. The disturbance—the "hump" of displaced string—travels *along* the string toward the bridge and nut, where it reflects and returns. At any moment you can see the displacement profile of the string: some segments are displaced above the resting position, some below, and the shape of this profile changes as time advances. But no piece of string travels from the plucking point to the bridge; each piece merely jiggles transversely in its own little neighbourhood.

Waves in which the oscillation is perpendicular to the direction of propagation are called **transverse waves**. The ripple on a still pond, electromagnetic radiation (light, radio waves, X-rays), and seismic S-waves are all transverse. They share the property that if you could label one particle of the medium and watch it, you would see it oscillate back and forth *across* the direction the wave is travelling, never along it.

2.3 Longitudinal Waves: The Displacement Is Parallel

Sound in air is fundamentally different. When a loudspeaker cone moves forward, it pushes the air molecules directly in front of it closer together, creating a small region of higher pressure (a *compression*). Those crowded molecules then push on their neighbours, and those on theirs, propagating the pressure disturbance forward. Between compressions, the speaker cone has pulled back, leaving a region of lower pressure (a *rarefaction*). The result is a series of alternating compressions and rarefactions—pressure crests and troughs—that marches outward from the speaker.

In this case, individual air molecules oscillate *back and forth along the direction of travel*—the same direction the wave is propagating. This is a **longitudinal wave**. Each air molecule moves a tiny distance—perhaps a tenth of a millimetre even for loud sound—but that local jiggling passes the disturbance on to the next layer of molecules with essentially no delay. The pattern of pressure variations travels at the speed of sound ($\approx 343\ \mathrm{m\,s^{-1}}$), while any individual molecule has an average speed that is far smaller.

Key distinction: what travels, and what does not.

In any wave:

- The **wave pattern** (shape, energy, information) travels forward at the wave speed v.
- The **medium particles** oscillate locally—transversely in a transverse wave, along the propagation direction in a longitudinal wave—and do not travel with the wave.
- After the wave has passed, each particle returns to its undisturbed position, as if the wave had never been. The medium is not permanently displaced by the wave; it is temporarily *borrowed* to carry the pattern.

This is why sound can cross a room without there being any bulk flow of air, why a wave on a rope does not carry the rope to the other end of the room, and why— most remarkably for this course—a sound wave that propagated for 380,000 years through the early universe left no net motion of matter behind, only a faint density imprint frozen at the moment the wave stopped.

2.4 The Phase: Keeping Track of Where You Are in the Oscillation

One more concept is essential before we write down the wave equation: the **phase**. Every oscillating particle in a wave is doing essentially the same thing—moving back and forth—but not all at the same time. A particle at position x_1 might be at the top of its oscillation while a particle at x_2 is at the bottom, and a particle at x_3 might be passing through the undisturbed position on its way up.

The *phase* of a particle's oscillation tells us where in its cycle it currently is: zero phase means it is at rest and moving in the positive direction, phase $\pi/2$ means it is at maximum positive displacement, phase π means it is back at rest but moving in the negative direction, and phase $3\pi/2$ means it is at maximum negative displacement. A full cycle returns the particle to phase $2\pi \equiv 0$.

For a simple sinusoidal wave travelling in the $+x$ direction, every particle oscillates as $y(x,t) = A\cos(kx - \omega t)$. The quantity in the argument, $kx - \omega t$, is the phase at position x and time t. Two particles separated by exactly one wavelength ($\Delta x = \lambda = 2\pi/k$) are always at the same phase and oscillate in lockstep. Two particles separated by half a wavelength are always exactly *out of phase*: when one is at its positive peak, the other is at its negative trough. Understanding phase is not just a mathematical nicety—the CMB acoustic peaks we will study in Lecture 6 arise precisely because different Fourier modes of the cosmic sound field were in different phases at the moment the universe became transparent.

3 The Vibrating String

With the conceptual picture firmly in place, we can now derive the mathematics. The vibrating string is our first quantitative model because it is simple enough to treat exactly, yet rich enough to reveal the essential character of all wave motion.

Consider a string of length L, mass per unit length μ (in $\mathrm{kg\,m^{-1}}$), under tension T (in Newtons). The string is stretched horizontally between two fixed supports. We pluck it slightly—just enough to create a small sideways displacement—and release it. What happens next?

3.1 Deriving the Wave Equation

To find the equation governing $y(x,t)$—the transverse displacement of the string at position x and time t—we isolate a tiny element of string between positions x and $x + \Delta x$ and apply Newton's second law to it.

The element has mass $\Delta m = \mu\,\Delta x$. The tension T acts along the string at both ends of the element; because the string is curved at the location of the element, the tension at the two ends points in slightly different directions, and these forces do not cancel exactly. Let $\theta(x)$ be the angle the string makes with the horizontal at position x. The upward component of tension at the right end of the element is $T\sin\theta(x+\Delta x) \approx T\tan\theta(x+\Delta x) = T\,(\partial y/\partial x)|_{x+\Delta x}$, and at the left end it is $-T\,(\partial y/\partial x)|_x$ (pointing downward at the left end). The approximation $\sin\theta \approx \tan\theta = \partial y/\partial x$ is valid as long as the displacement is small—the *small-amplitude approximation*— which we will maintain throughout.

The net upward force on the element is therefore:

$$\Delta F_y \approx T\,\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - T\,\frac{\partial y}{\partial x}\bigg|_{x} = T\,\frac{\partial^2 y}{\partial x^2}\,\Delta x, \tag{1}$$

where the last step uses the definition of the second derivative: the change in $\partial y/\partial x$ over an interval Δx is $(\partial^2 y/\partial x^2)\Delta x$.

Newton's second law, $\Delta F_y = \Delta m \cdot a_y$, then gives:

$$T\,\frac{\partial^2 y}{\partial x^2}\,\Delta x = \mu\,\Delta x\,\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

Dividing both sides by Δx, which cancels, we arrive at the **one-dimensional wave equation**:

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2\,\frac{\partial^2 y}{\partial x^2}}, \qquad v = \sqrt{\frac{T}{\mu}}. \tag{3}$$

This is one of the most important equations in physics. It says that the *acceleration* of the string at any point (the left side) is proportional to the *curvature* of the string at that point (the right side). Intuitively, a sharply curved section of string has the tension forces pulling it more strongly back toward horizontal; a flat section feels no net restoring force. Curvature drives acceleration: that is the entire physics of the wave equation.

The quantity v is the speed at which disturbances travel along the string. It increases with tension T (a tighter string is stiffer and responds more forcefully to any curvature, accelerating each element back toward equilibrium faster) and decreases with linear density μ (a heavier string is more sluggish, requiring more force to achieve the same acceleration). The formula $v = \sqrt{T/\mu}$ is the mathematical expression of the intuition that "stiffness over inertia determines wave speed"—a principle we will use repeatedly.

Numerical example: a guitar E-string. The low-E string on a concert guitar has linear density $\mu = 5.0 \times 10^{-3}\ \mathrm{kg\,m^{-1}}$, length $L = 0.65$ m, and is tuned to $f_1 = 82.4$ Hz (standard pitch, with $A_4 = 440$ Hz). What tension is required?

The fundamental mode has both ends fixed, so it fits exactly half a wavelength in the string: $\lambda_1 = 2L = 1.30$ m. The wave speed must therefore be:

$$v = \lambda_1 f_1 = 1.30 \times 82.4 = 107\ \mathrm{m\,s^{-1}}.$$

From $v = \sqrt{T/\mu}$ we get:

$$T = \mu v^2 = 5.0 \times 10^{-3} \times (107)^2 = 5.0 \times 10^{-3} \times 11{,}449 \approx 57 \text{ N}.$$

This is equivalent to about 5.8 kg hanging from the string—entirely consistent with the 50–80 N tensions found in guitar construction. Notice how the formula links the audible frequency directly to the physical properties of the string.

3.2 Standing Waves and Normal Modes

The wave equation (3) has infinitely many solutions. For a string fixed at $x = 0$ and $x = L$, the boundary conditions $y(0,t) = y(L,t) = 0$ select a discrete family. To find them, we try *separable* solutions of the form $y(x,t) = X(x)\,T_{\text{time}}(t)$—that is, functions that factorise into a purely spatial part and a purely temporal part. Substituting into (3):

$$X\,\ddot{T}_{\text{time}} = v^2\,X''\,T_{\text{time}} \quad \Rightarrow \quad \frac{\ddot{T}_{\text{time}}}{T_{\text{time}}} = v^2\,\frac{X''}{X}.$$

The left side depends only on t; the right side depends only on x. The only way this can hold for all x and t simultaneously is if both sides equal the same constant. We call it $-\omega^2$ (negative so that the time part oscillates rather than grows or decays):

$$\ddot{T}_{\text{time}} = -\omega^2 T_{\text{time}}, \qquad X'' = -k^2 X, \quad k = \frac{\omega}{v}.$$

The time equation has the general solution $T_{\text{time}}(t) = C\cos(\omega t) + D\sin(\omega t)$: pure oscillation at frequency ω. The spatial equation has the general solution $X(x) = A\sin kx + B\cos kx$. Applying the boundary condition at $x = 0$ forces $B = 0$; applying it at $x = L$ forces $\sin kL = 0$, which is satisfied only when $kL = n\pi$ for a positive integer $n = 1, 2, 3, \ldots$ The allowed wavenumbers are therefore:

$$k_n = \frac{n\pi}{L}, \qquad \lambda_n = \frac{2L}{n}, \qquad f_n = \frac{v}{\lambda_n} = \frac{nv}{2L}.$$

These are the **harmonic frequencies**. The $n = 1$ mode (the *fundamental*) has wavelength $\lambda_1 = 2L$ and a single arch of displacement filling the string. The $n = 2$ mode (*first overtone*) has $\lambda_2 = L$, with two arches and a *node*—a point of zero displacement—in the middle. The $n = 3$ mode has three arches, two nodes, and so on. Each mode is a pure sinusoid in space that oscillates in time at its own frequency; these are the **normal modes** of the string.

The general motion of the string is a superposition of all these modes:

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\!\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \phi_n). \tag{4}$$

This is a physical realisation of a Fourier series: any initial shape can be decomposed into sinusoidal building blocks. A plucked guitar string rings at the fundamental and its overtones simultaneously, which is precisely what gives the instrument its characteristic timbre.

Crucially, after the wave has passed—after the guitar string has been allowed to come to rest—every piece of string returns to exactly where it was. The pattern has propagated back and forth many times, but no material has been permanently relocated. This is the stadium-wave principle in action.

4 Sound in Air: Pressure Waves

A guitar string perturbs the air around it, and the air carries the disturbance to our ears. But the mechanism is quite different from what we just studied. The string undergoes *transverse* oscillation—its particles move perpendicular to the string. Sound in air involves *longitudinal* oscillation—the air molecules move *back and forth along the direction the sound travels.*

Imagine the air as an enormous collection of molecules, each rattling around rapidly in random thermal motion. When a sound wave passes, it superimposes a tiny organised back-and-forth motion on top of that random agitation. The organised motion is a small fraction of the thermal motion—for a normal conversational sound, the organised displacement amplitude is roughly a tenth of a millimetre, while the thermal speed of air molecules at room temperature is about $500\,\mathrm{m\,s^{-1}}$. The wave is a very gentle nudge on top of a great deal of background noise.

What propagates is not the molecules themselves but a *pattern of pressure variation*: alternating regions where the molecules are squeezed closer together (compressions, high pressure) and regions where they are pulled further apart (rarefactions, low pressure). Each molecule responds to its local pressure gradient, accelerating toward the low-pressure region. In doing so, it transmits the pressure variation to its neighbours, and those transmit it to theirs—the pattern propagates. Each molecule eventually returns to its undisturbed position once the wave has passed.

4.1 The Bulk Modulus

The key material property that governs how a gas responds to compression is the **bulk modulus** B. It is defined as the pressure increase produced by a fractional decrease in volume:

$$B = -V\,\frac{\partial P}{\partial V} = \rho\,\frac{\partial P}{\partial \rho}. \tag{5}$$

A large B means the medium strongly resists compression—it is "stiff" to pressure waves. Steel has $B \approx 160$ GPa; water has $B \approx 2.2$ GPa; air at atmospheric pressure has $B \approx 142$ kPa. The enormous difference in bulk modulus between solids and gases explains why sound travels so much faster in steel ($\approx 5100\ \mathrm{m\,s^{-1}}$) than in air ($\approx 343\ \mathrm{m\,s^{-1}}$)—the higher stiffness more than compensates for the higher density.

For an *isothermal* process in an ideal gas, $PV = \text{const}$, so $P\,dV + V\,dP = 0$, giving $dP/dV = -P/V$ and $B_{\text{iso}} = P$. But Laplace's correction (Section 1) tells us that sound compressions are *adiabatic*. For an adiabatic process, $PV^{\gamma} = \text{const}$, differentiating gives $dP/dV = -\gamma P/V$, and therefore:

$$B_{\text{ad}} = \gamma P. \tag{6}$$

The adiabatic bulk modulus is $\gamma \approx 1.4$ times larger than the isothermal one: adiabatic compression is stiffer because the temperature rises, providing extra pressure beyond what isothermal compression would.

4.2 Deriving the Speed of Sound

We can derive the speed of sound by the same "stiffness over inertia" reasoning used for the string, replacing tension with bulk modulus and linear density with volume density. Carrying out the derivation carefully—applying conservation of mass (continuity equation) and Newton's second law to a small fluid parcel, using the adiabatic bulk modulus for the pressure–density relationship—yields:

$$\boxed{c_{\text{sound}} = \sqrt{\frac{B}{\rho}} = \sqrt{\frac{\gamma P}{\rho}}}. \tag{7}$$

For an ideal gas at temperature T_{K} (in Kelvin), the equation of state gives $P = \rho R_{\mathrm{sp}} T_{\mathrm{K}}$, where $R_{\mathrm{sp}} = R/M_{\mathrm{air}} = 8.314/0.029 \approx 287\ \mathrm{J\,kg^{-1}K^{-1}}$ is the specific gas constant for air. Substituting:

$$c_{\mathrm{sound}} = \sqrt{\frac{\gamma R_{\mathrm{sp}} T_{\mathrm{K}}}{1}} = \sqrt{\gamma R_{\mathrm{sp}} T_{\mathrm{K}}}. \tag{8}$$

The speed of sound in air depends only on temperature—not on pressure or density separately. Doubling the pressure at constant temperature doubles the density too (ideal gas law), so the ratio P/ρ stays fixed. This is a non-obvious but very important result: at a given temperature, every ideal gas has the same speed of sound (once you account for the different γ and molecular mass). And since $c_{\mathrm{sound}} \propto \sqrt{T_{\mathrm{K}}}$, sound travels faster in warm air than in cold—the reason why orchestra instruments go sharp as the concert hall heats up.

Numerical example: speed of sound at room temperature. At $T_{\mathrm{K}} = 293$ K (20°C), with $\gamma = 1.40$ for diatomic air:

$$c_{\mathrm{sound}} = \sqrt{1.40 \times 287 \times 293} = \sqrt{117{,}776} \approx 343\ \mathrm{m\,s^{-1}}.$$

At 0°C: $c_{\mathrm{sound}} = \sqrt{1.40 \times 287 \times 273} = \sqrt{109{,}754} \approx 331\ \mathrm{m\,s^{-1}}$.

Newton's error. Without Laplace's correction, using $B = P$ (isothermal, $\gamma = 1$): $c_{\mathrm{Newton}} = \sqrt{287 \times 293} = \sqrt{84{,}091} \approx 290\ \mathrm{m\,s^{-1}}$. The ratio $343/290 = 1.183 = \sqrt{1.40} = \sqrt{\gamma}$: Newton's prediction is off by 18%, precisely as Laplace diagnosed.

5 Resonance in Pipes

Having established how sound propagates through a continuous medium, we can now apply the same boundary-condition logic we used for the string to understand resonance in organ pipes.

The key difference is that we are now dealing with a *longitudinal* wave. For the string, we tracked transverse displacement; for the pipe, we can track either the *pressure variation* $\delta P(x,t)$ or the *displacement* $s(x,t)$ of air parcels along the pipe. These two descriptions are related but have opposite boundary conditions at the ends of the pipe:

- At a **closed end**, air parcels cannot move (there is a rigid wall). The displacement s has a *node* (zero) there. But the pressure variation is maximum: the closed end is a pressure *antinode*. This is analogous to a fixed end on a string (displacement node).
- At an **open end**, the pressure is fixed at atmospheric (any excess pressure escapes freely). The pressure variation δP has a *node* there. But the air is free to move, so the displacement has an *antinode*.

For a pipe of length L **open at both ends**, the displacement field has antinodes at both ends (just like the string, which has displacement antinodes in the interior but—here is the subtlety—the open pipe's displacement pattern matches the *pressure* pattern of a closed string). Working through the boundary conditions gives the same set of allowed frequencies as the doubly-fixed string:

$$f_n = \frac{n\, c_{\mathrm{sound}}}{2L}, \qquad n = 1, 2, 3, \ldots$$

All harmonics—fundamental, octave, fifth above octave, etc.—are present.

For a pipe **closed at one end and open at the other**, the displacement has a node at the closed end and an antinode at the open end. The lowest-frequency mode fits one quarter-wavelength in the pipe: $\lambda_1 = 4L$. Only *odd* harmonics are present:

$$f_n = \frac{(2n-1)\,c_{\text{sound}}}{4L}, \qquad n = 1, 2, 3, \ldots$$

Why a clarinet sounds an octave lower than a flute of the same length. A flute is open at both ends and behaves like a doubly-open pipe; its fundamental frequency is $c_{\text{sound}}/(2L)$. A clarinet is approximately a cylindrical pipe closed at the mouthpiece end (the reed acts as a pressure antinode, which is a displacement node), so its fundamental frequency is $c_{\text{sound}}/(4L)$—exactly half the flute's, i.e. one octave lower. A flute and clarinet of the same physical length therefore play in registers an octave apart. Furthermore, the clarinet produces only odd harmonics ($n = 1, 3, 5, \ldots$), which gives its tone a hollow, woody quality very different from the flute's bright, full-harmonic sound. These differences arise entirely from the boundary condition at one end of a cylindrical tube.

Numerical example: the lowest note of an organ pipe. A large open organ pipe has $L = 4.88$ m (the "16-foot pipe", by the old measurement). At $c_{\text{sound}} = 343\ \mathrm{m\,s^{-1}}$:

$$f_1 = \frac{343}{2 \times 4.88} = \frac{343}{9.76} \approx 35.1\ \text{Hz}.$$

This is close to C_1 (32.7 Hz) on the piano—near the lower threshold of human hearing. The pipe resonates because the round-trip travel time for a pressure pulse—from one end to the other and back—equals exactly one period of oscillation: $2L/c_{\text{sound}} = 9.76/343 \approx 0.0285$ s, and $1/f_1 = 1/35.1 \approx 0.0285$ s. Resonance is simply repeated constructive interference: pulses that have bounced back arrive in step with the next outgoing pulse, and they reinforce each other.

6 Fourier Analysis: Every Shape Is a Sum of Sinusoids

We have now seen standing modes on a string and in a pipe. In each case, the general motion is a superposition of discrete sinusoidal modes. But where does this idea lead us when we want to describe not a standing pattern but an *arbitrary* wave shape—say, the initial displaced profile of a plucked guitar string, or the primordial density fluctuations generated by inflation? This is where Fourier analysis enters, and it is one of the most powerful tools in all of physics.

6.1 The Fourier Series

Joseph Fourier (1768–1830) worked on the problem of heat conduction in a metal bar during Napoleon's Egyptian campaign, in which he participated as a scientific administrator. His central mathematical claim, published in *Théorie analytique de la chaleur* (1822), was audacious for its time: that *any* function defined on a finite interval, no matter how jagged or discontinuous, can be expressed as an infinite sum of sine and cosine functions.

The claim provoked fierce debate. Lagrange (who had done important work on vibrating strings), Laplace, and Poisson all had reservations about whether a sum of smooth, infinitely-differentiable sinusoids could really represent a function with a corner or a jump. The mathematicians were right to be cautious—the precise conditions of convergence took decades to work out fully, involving the work of Dirichlet, Riemann, and eventually Cantor's theory of infinite sets. But the *physics* worked immediately: Fourier's method gave correct answers for heat flow, and it was quickly adopted by physicists for wave problems.

The Fourier series for a function $f(x)$ on the interval $[0, L]$ is:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{9}$$

where the coefficients are obtained by exploiting the fact that sines and cosines of different frequencies are *orthogonal*: their product integrates to zero over the full interval. Specifically:

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{10}$$

The orthogonality can be verified directly: for $m \neq n$, $\int_0^L \sin(m\pi x/L)\sin(n\pi x/L)\,dx = 0$, while the self-integral $\int_0^L \sin^2(n\pi x/L)\,dx = L/2$. These are just integral identities for products of sinusoids, and they are the reason the coefficients b_n can be extracted cleanly by the integral formula above without the different modes "bleeding into" one another.

For an extended medium, the sum over discrete modes becomes an integral: the **Fourier transform**.

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx} \frac{dk}{2\pi}, \qquad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx. \tag{11}$$

Here $\tilde{f}(k)$ is the **Fourier amplitude** at wavenumber k: it tells us how much of spatial frequency k is present in $f(x)$. The **power spectrum** $|\tilde{f}(k)|^2$ measures the energy contributed by each wavenumber. When cosmologists study the CMB temperature fluctuations or the galaxy density field, they are computing exactly this—decomposing the sky into its Fourier modes and asking how much variance exists at each angular scale.

6.2 Why Fourier Analysis and Waves Belong Together

The deep connection between Fourier analysis and wave physics comes from the fact that the wave equation is *linear*: if $y_1(x,t)$ and $y_2(x,t)$ are both solutions, so is $y_1 + y_2$. This means we can build up any solution from simpler ones. The simplest solutions of the wave equation are pure sinusoids: $e^{i(kx-\omega t)}$ is a solution for any k, provided $\omega = vk$.

If the medium is linear and the initial shape is $y(x,0) = f(x)$, we decompose $f(x)$ into its Fourier modes $\tilde{f}(k)$, let each mode evolve independently (it simply picks up the phase factor $e^{-i\omega t}$), and reassemble the result:

$$y(x,t) = \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{i(kx-\omega(k)t)} \frac{dk}{2\pi}. \tag{12}$$

For a non-dispersive medium ($\omega = vk$), this becomes $y(x,t) = f(x - vt)$: the entire initial profile translates rigidly at speed v without changing shape. For a *dispersive* medium—one where ω is not proportional to k— different wavenumbers travel at different speeds, and the pulse spreads out and distorts as it propagates. We will encounter dispersion extensively in Lecture 2.

Numerical example: Fourier coefficients of a plucked string. A string of length $L = 1$ m is plucked at its midpoint, giving an initial triangular displacement:

$$y(x,0) = \begin{cases} 2Ax/L & 0 \le x \le L/2, \\ 2A(L-x)/L & L/2 \le x \le L, \end{cases}$$

with $A = 5$ mm. The Fourier sine coefficients are:

$$b_n = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

For odd n: $b_1 = 8A/\pi^2$, $b_3 = -8A/(9\pi^2)$, $b_5 = 8A/(25\pi^2)$, $\ldots$; for even n, $b_n = 0$ (modes with a node at the midpoint receive no energy from a midpoint pluck). Substituting $A = 5$ mm:

$$b_1 = \frac{8\times 5}{9.87} \approx 4.05 \text{ mm}, \quad b_3 \approx -0.45 \text{ mm}, \quad b_5 \approx 0.16 \text{ mm}.$$

The fundamental mode carries $(4.05)^2/(4.05^2 + 0.45^2 + 0.16^2 + \cdots) \approx 98.8\%$ of the energy—which is why a midpoint pluck produces a nearly pure tone. Plucking near the bridge excites high harmonics strongly, producing the bright, twangy sound of flamenco guitar.

7 Three-Dimensional Waves and Spherical Symmetry

So far we have confined ourselves to one dimension. Real sound fills space in three dimensions: when a door slams or a star explodes, the disturbance spreads outward in a sphere. The three-dimensional wave equation for a pressure perturbation $p(\mathbf{r}, t)$ is:

$$\frac{\partial^2 p}{\partial t^2} = c_s^2\,\nabla^2 p, \tag{13}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian. This admits plane-wave solutions $p \propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ with $\omega = c_s|\mathbf{k}|$, just as in one dimension.

For a source with spherical symmetry—a firecracker, a supernova, or the primordial quantum seed of cosmic structure—we write $p = p(r, t)$ where $r = |\mathbf{r}|$. The Laplacian in spherical coordinates for a spherically symmetric function is $\nabla^2 p = r^{-1}\,\partial^2(rp)/\partial r^2$, so equation (13) becomes:

$$\frac{\partial^2(rp)}{\partial t^2} = c_s^2\,\frac{\partial^2(rp)}{\partial r^2}.$$

This is exactly the 1D wave equation for the product $u(r,t) \equiv r\,p(r,t)$. Its general solution is $u = f(r - c_s t) + g(r + c_s t)$, giving:

$$p(r,t) = \frac{f(r - c_s t)}{r} + \frac{g(r + c_s t)}{r}. \tag{14}$$

The first term is an outgoing spherical wave: a disturbance that was created at $r = 0$ at time $t = 0$ and now occupies a thin shell at radius $r = c_s t$. The crucial factor is $1/r$: the pressure amplitude falls as the inverse of distance. Since intensity $I \propto p^2$, the intensity falls as $1/r^2$—the familiar **inverse-square law**, here derived as a direct consequence of the spreading of energy over a surface of area $4\pi r^2$.

This spherical-wave picture is precisely what we will use in Lecture 5. A quantum fluctuation in the infant universe creates a tiny overdensity at some point in space. That overdensity excites an outgoing spherical pressure wave in the photon–baryon plasma, which expands at the cosmic sound speed for 380,000 years. When the universe becomes transparent (recombination), the wave freezes: it can no longer propagate because photon pressure has been removed. It leaves a faint circular ring—a slight overdensity of matter—at a fixed comoving radius of 150 Mpc. Repeated over billions of initial seeds, these rings accumulate into a statistical excess in the separation between galaxies at exactly that scale. This is the BAO feature. Its mathematical ancestor is the simple equation $r = c_s t$.

Numerical example: inverse-square law for thunder. A lightning bolt at distance $r_1 = 1$ km produces a sound pressure level $L_1 = 100$ dB ($p_1 = 2$ Pa rms). What is the level at $r_2 = 5$ km?

The pressure amplitude scales as $p \propto 1/r$, so $p_2/p_1 = r_1/r_2 = 1/5$. The sound level in decibels is $L = 20 \log_{10}(p/p_0)$, so:

$$L_2 - L_1 = 20 \log_{10}(1/5) = 20 \times (-0.699) = -14.0 \text{ dB}.$$

The thunder at 5 km is heard at 86 dB—still clearly audible. Each doubling of distance reduces the level by $20 \log_{10} 2 = 6$ dB.

8 Phase Velocity and Group Velocity

All the waves we have studied so far have been *non-dispersive*: the wave speed v is the same for all frequencies. The wave equation directly implies $\omega = vk$ (a linear dispersion relation), and all wavelengths travel at the same speed. This means that the initial shape of a wave pulse travels unchanged: as we showed in equation (12), the solution is simply $f(x - vt)$.

Many real media are *dispersive*: the wave speed depends on frequency. Ocean surface waves, light in glass, quantum matter waves—and, crucially for this course, sound in the photon–baryon plasma of the early universe—are all dispersive. In such media, different Fourier components of a pulse travel at different speeds, so the pulse spreads out and distorts as it propagates. To handle dispersion we need two distinct velocity concepts.

8.1 The Dispersion Relation

A **dispersion relation** $\omega(k)$ describes how the angular frequency $\omega = 2\pi f$ depends on wavenumber $k = 2\pi/\lambda$ for a given medium. For a non-dispersive medium, $\omega = vk$ is a straight line through the origin. For a dispersive medium the curve bends, and different k-values lie on different parts of the curve.

Consider a discrete chain of masses connected by springs—a simple model for a crystalline solid. The exact dispersion relation is $\omega = \omega_{\max}|\sin(ka/2)|$, where a is the lattice spacing. For small k (long wavelengths, many atoms per wavelength) this is nearly linear: $\omega \approx (\omega_{\max}\, a/2)\, k$, and the chain behaves like a non-dispersive continuous medium. But for large k (short wavelengths, comparable to the lattice spacing), the dispersion relation saturates at $\omega_{\max}$—the wave cannot oscillate faster than the maximum rate at which springs can push masses. We will see an analogous saturation in the cosmic plasma in Lecture 3.

8.2 Phase Velocity

The **phase velocity** v_{ph} is the speed at which a surface of constant phase advances. For a pure sinusoidal wave $p = A\cos(kx - \omega t)$, a crest (point of constant phase) moves to keep $kx - \omega t = \text{const}$, so $k\,dx = \omega\,dt$ and:

$$v_{\text{ph}} = \frac{\omega}{k}. \tag{15}$$

For a non-dispersive medium, $\omega/k = v$ for all k, and all crests travel at the same speed. For a dispersive medium, different Fourier components have different phase velocities, causing the pulse to spread as the faster components race ahead of the slower ones.

8.3 Group Velocity

A real pulse is not a pure sinusoid but a superposition of many Fourier components concentrated near some central wavenumber k_0. Consider the simplest possible case: two waves of nearly equal wavenumber, k and $k + \delta k$:

$$y = \cos(kx - \omega t) + \cos\big[(k + \delta k)x - (\omega + \delta\omega)t\big].$$

Using the sum-to-product identity $\cos A + \cos B = 2\cos\big(\frac{A-B}{2}\big)\cos\big(\frac{A+B}{2}\big)$:

$$y = 2\cos\left(\frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right)\cos\left(kx - \omega t + \frac{\delta k}{2}x - \frac{\delta\omega}{2}t\right).$$

For small δk, the second cosine oscillates rapidly at roughly the original frequency—this is the **carrier wave**, the individual crests visible within the pulse. The first cosine oscillates much more slowly in space and time—this is the **envelope**, the overall shape of the pulse. The carrier moves at the phase velocity ω/k; the *envelope* moves at $\delta\omega/\delta k$. Taking the limit:

$$\boxed{v_{\text{gr}} = \frac{d\omega}{dk}.} \tag{16}$$

The **group velocity** v_{gr} is the speed of the envelope—the speed at which the pulse as a whole advances, and hence the speed at which energy and information travel. For a non-dispersive medium, $\omega = vk$ gives $v_{\text{gr}} = v = v_{\text{ph}}$; the two velocities are equal and both equal to the wave speed.

For a dispersive medium, v_{gr} and v_{ph} can differ dramatically, and it is the group velocity that controls physical transport. In Lecture 2 we will encounter a dispersion relation $\omega^2 = c^2k^2 + \omega_0^2$ (the Klein–Gordon relation, governing light in a plasma), for which the phase velocity *exceeds* c but the group velocity is *below* c. There is no contradiction: the phase velocity is the speed of a mathematical crest—a surface of constant phase—not of any physical object or signal.

Numerical example: deep-water gravity waves. Ocean waves in deep water obey $\omega = \sqrt{gk}$, where $g = 9.8\ \text{m}\,\text{s}^{-2}$. For a wave of wavelength $\lambda = 100$ m, $k = 2\pi/100 = 0.0628\ \text{rad}\,\text{m}^{-1}$:

$$\omega = \sqrt{9.8 \times 0.0628} = \sqrt{0.615} = 0.785\ \text{rad}\,\text{s}^{-1}, \quad f = 0.125\ \text{Hz}, \quad T = 8\ \text{s}.$$

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{0.785}{0.0628} \approx 12.5\ \text{m}\,\text{s}^{-1}.$$

$$v_{\text{gr}} = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{gk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_{\text{ph}}}{2} \approx 6.25\ \text{m}\,\text{s}^{-1}.$$

The group velocity is *half* the phase velocity. This is why, watching a group of ocean swells from above, individual crests appear to emerge at the rear of the group, travel forward through it, and disappear at the front—they move twice as fast as the group. The group (and its energy) advances at $6.25\,\mathrm{m\,s^{-1}}$ while individual crests advance at $12.5\,\mathrm{m\,s^{-1}}$. It looks like an illusion, but it is a direct consequence of $v_{\mathrm{gr}} = v_{\mathrm{ph}}/2$.

9 Energy in a Wave

Waves exist to transport energy. A plucked string has energy; a sound wave carries energy from a loudspeaker to your ears; electromagnetic waves carry the sun's energy across 150 million kilometres of vacuum. Understanding how energy is stored and transported in a wave is essential for tracking the energy budget of the early universe.

For a one-dimensional wave $y = A\cos(kx - \omega t)$ on a string, the **kinetic energy per unit length** is:

$$\mathcal{K} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu\,\omega^2 A^2 \sin^2(kx - \omega t).$$

(Each element $\mu\,dx$ has velocity $\partial y/\partial t = -A\omega\sin(kx - \omega t)$, and kinetic energy $\frac{1}{2}(\mu\,dx)\,v^2$.) The **potential (elastic) energy per unit length** stored in the curvature of the string is:

$$\mathcal{U} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2}T\,k^2 A^2 \sin^2(kx - \omega t).$$

Since $Tk^2 = \mu\omega^2$ (from $v = \sqrt{T/\mu}$ and $\omega = vk$), the kinetic and potential energy densities are always *equal*. This is not a coincidence: it is the virial theorem for a harmonic oscillator, and it holds for all wave equations of the form (3). The time-averaged total energy per unit length is:

$$\langle \mathcal{E} \rangle = \frac{1}{2}\mu\,\omega^2 A^2. \tag{17}$$

Energy is proportional to the *square* of the amplitude and to the *square* of the frequency. Doubling the amplitude quadruples the energy; doubling the frequency quadruples the energy. This square-law is universal in wave physics—it is the reason why the power spectrum in cosmology is expressed as the square of the amplitude of density fluctuations.

9.1 Energy Flux and Intensity

Energy does not merely reside in the wave; it moves. The rate at which energy passes a fixed cross-section per unit area is the **intensity** I (in $\mathrm{W\,m^{-2}}$). For a plane sound wave in a medium of density ρ_0 and sound speed c_s, the intensity is:

$$I = \frac{p_0^2}{2\rho_0 c_s}, \tag{18}$$

where p_0 is the pressure amplitude. The combination $Z = \rho_0 c_s$ is the **acoustic impedance**. When a sound wave crosses an interface between two media of different impedances, the fraction of intensity reflected is:

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^2. \tag{19}$$

This impedance-matching condition is precisely why medical ultrasound requires a gel between the transducer and the skin—without it, the enormous mismatch between air ($Z \approx 415\ \mathrm{Pa\,s\,m^{-1}}$) and tissue ($Z \approx 1.5 \times 10^6\ \mathrm{Pa\,s\,m^{-1}}$) would reflect nearly 100% of the ultrasound signal before it even entered the body.

Numerical example: threshold of pain. The threshold of human hearing corresponds to intensity $I_0 = 10^{-12}$ W m^{-2} and pressure amplitude $p_{\text{th}} = 2 \times 10^{-5}$ Pa. The threshold of pain is $I_{\text{pain}} = 1$ W m^{-2}.

From equation (18), $I \propto p_0^2$ at fixed $\rho_0 c_s$, so:

$$\frac{p_{\text{pain}}}{p_{\text{th}}} = \sqrt{\frac{I_{\text{pain}}}{I_0}} = \sqrt{\frac{1}{10^{-12}}} = 10^6.$$

Thus $p_{\text{pain}} = 10^6 \times 2 \times 10^{-5} = 20$ Pa. Atmospheric pressure is 10^5 Pa, so even a damaging sound wave is only a 0.02% perturbation of background pressure. Sound is always a *small perturbation*. The same will be true of the primordial density fluctuations ($\delta\rho/\rho \sim 10^{-5}$), which is why linear perturbation theory works so well in Lectures 4 and 5.

9.2 The Equipartition Theorem and a Preview of Radiation Physics

There is a beautiful connection between wave energy and thermodynamics that foreshadows the treatment of thermal radiation in Lecture 3. For a system in thermal equilibrium at temperature T_{K}, the **equipartition theorem** assigns average energy $\frac{1}{2}k_{\text{B}}T_{\text{K}}$ to each independent quadratic degree of freedom, where $k_{\text{B}} = 1.38 \times 10^{-23}$ J K^{-1} is Boltzmann's constant. A standing-wave mode of a string has one kinetic degree of freedom ($\propto \dot{q}^2$) and one potential degree of freedom ($\propto q^2$), so it receives average thermal energy $k_{\text{B}}T_{\text{K}}$.

Summing over all modes of a three-dimensional cavity leads, in the classical limit, to the **Rayleigh–Jeans law** for thermal radiation—and famously to the "ultraviolet catastrophe": because the number of modes increases without limit at high frequency, the classical prediction is that a cavity should radiate infinite total energy. This absurd conclusion motivated Max Planck to introduce energy quantisation in 1900, and the correct distribution—the Planck spectrum— follows directly once one assumes that each mode of frequency ν can only carry energy in discrete lumps of $h\nu$. The high-frequency modes are then exponentially "frozen out" at temperatures too low to excite a single quantum. This leads directly to the concept of photons, which are the dominant energy component of the early universe. We will not pursue this further here, but it is worth holding in mind: the physics of a vibrating string, taken seriously, contains within it the seeds of quantum mechanics.

10 Summary and the Road Ahead

We have covered substantial ground. Starting from a vibrating string, we derived the wave equation and found that a medium's wave speed is set by the ratio of its restoring force (stiffness) to its inertia. For a string those are tension and linear density; for a gas they are the adiabatic bulk modulus and the mass density. We introduced the dispersion relation $\omega(k)$ and separated the phase velocity—the speed of individual crests—from the group velocity—the speed of energy and information. We proved that energy in a wave is proportional to the square of the amplitude and frequency, and we saw how three-dimensional spherical waves obey an inverse-square law. Finally, Fourier analysis—the decomposition of any waveform into sinusoidal modes—emerged as the unifying language connecting all these ideas.

Above all, we established the crucial conceptual divide between *what travels* (the wave pattern, the energy, the phase information) and *what does not travel* (the medium particles, which merely oscillate and return). This distinction will be tested and deepened at every stage: in the cosmic plasma, it is the acoustic wave pattern—not any particular photon or proton— that propagates for 380,000 years, and it is the frozen imprint of that pattern, not any motion of matter, that we observe as the BAO feature today.

Tables 1 and 2 collect the key numerical results and conceptual connections of this lecture.

System	**Formula**	**Numerical value**
Guitar low-E string	$v = \sqrt{T/\mu}$	$107\ \mathrm{m\,s^{-1}}$
Sound in air, 20°C	$c_s = \sqrt{\gamma R_{\mathrm{sp}} T_{\mathrm{K}}}$	$343\ \mathrm{m\,s^{-1}}$
Sound in air, 0°C	same formula	$331\ \mathrm{m\,s^{-1}}$
Newton (isothermal, 0°C)	$c = \sqrt{P/\rho}$	$290\ \mathrm{m\,s^{-1}}$
Ocean wave ($\lambda = 100$ m) phase	$v_{\mathrm{ph}} = \sqrt{g/k}$	$12.5\ \mathrm{m\,s^{-1}}$
Ocean wave ($\lambda = 100$ m) group	$v_{\mathrm{gr}} = \frac{1}{2}\sqrt{g/k}$	$6.25\ \mathrm{m\,s^{-1}}$

Table 1: Wave speeds derived or computed in Lecture 1.

Concept (Lecture 1)	**Cosmic counterpart (Lectures 3–7)**
Wave pattern travels; particles stay	Acoustic shell expands; matter returns
Bulk modulus $B = \gamma P$	Radiation pressure $P_{\mathrm{rad}} = aT^4/3$
Linear density μ	Photon–baryon fluid density $\rho_\gamma + \rho_b$
Speed of sound $c_s = \sqrt{B/\rho}$	Cosmic sound speed $c_s = c/\sqrt{3(1+R_b)}$
Standing modes $k_n = n\pi/L$	Acoustic peaks in the CMB power spectrum
Spherical wave $p \propto 1/r$	Acoustic shell at the BAO scale
Fourier power spectrum $\lvert\tilde{f}(k)\rvert^2$	CMB C_ℓ and matter power spectrum $P(k)$
Group velocity carries energy	Sound horizon $r_s = \int c_s\, d\eta$ freezes at decoupling

Table 2: Lecture 1 concepts and their counterparts in the cosmic plasma.

In **Lecture 2** we will complicate the medium. We will consider fluids composed of multiple components with different densities and compressibilities, find that the wave equation acquires coupling terms between them, and discover the Jeans instability—the tipping point at which a self-gravitating medium cannot sustain a sound wave and instead collapses. This is gravity's first intervention in our story, and it is the mechanism that will turn the cosmos's acoustic oscillations into galaxies. The concepts of phase and group velocity mastered here will return when we encounter the plasma cut-off frequency in Lecture 3, and again when we track the propagation of the cosmic sound horizon in Lecture 5.

"The laws of nature are but the mathematical thoughts of God."
—Attributed to Euler; fitting nonetheless.

Exercises

Solutions to odd-numbered exercises will be discussed at the beginning of Lecture 2.

1. A steel piano wire has length $L = 0.40$ m, diameter $d = 0.90$ mm, and density $\rho_{\text{steel}} = 7800\ \text{kg}\,\text{m}^{-3}$. (a) Compute the linear density μ in $\text{kg}\,\text{m}^{-1}$. (b) If the wire is to produce the note $A_4 = 440$ Hz, what tension is required? (c) If the tension is doubled, by what factor does the frequency change? (d) If the wire length is doubled (at the same tension and material), what happens to the fundamental frequency?

2. A thin rod has a closed (displacement node) end at $x = 0$ and an open (displacement antinode) end at $x = L$. (a) Sketch the displacement mode shapes for $n = 1, 2, 3$. (b) Derive the allowed wavelengths λ_n and frequencies f_n. (c) A particular organ pipe of this type has $L = 0.58$ m at $T = 20°\text{C}$. Compute f_1, f_2, and f_3, and state which is the fundamental. (d) Explain physically why the even harmonics (in the open-open numbering) are missing.

3. The speed of sound in seawater is approximately $1530\ \text{m}\,\text{s}^{-1}$ and the density of seawater is $1025\ \text{kg}\,\text{m}^{-3}$. (a) What is the bulk modulus of seawater? (b) Compare the acoustic impedance of seawater with that of air ($\rho_{\text{air}} = 1.2\ \text{kg}\,\text{m}^{-3}$, $c_{\text{air}} = 343\ \text{m}\,\text{s}^{-1}$). (c) Using equation (19), what fraction of sound intensity is reflected when a wave in air hits a flat water surface perpendicularly? (d) Comment on what this means for underwater communication.

4. A dispersion relation has the form $\omega = \alpha k^2$, where α is a constant with units $\text{m}^2\,\text{s}^{-1}$. (a) Find $v_{\text{ph}}(k)$ and $v_{\text{gr}}(k)$. (b) Is v_{gr} greater or less than v_{ph}? (c) For a pulse centred at $k_0 = 10\ \text{rad}\,\text{m}^{-1}$ with $\alpha = 1\ \text{m}^2\text{s}^{-1}$, compute both velocities. (d) This dispersion relation also governs quantum matter waves (de Broglie waves), where $\alpha = \hbar/(2m)$. For an electron ($m = 9.11 \times 10^{-31}$ kg) with de Broglie wavelength $\lambda = 1$ nm, compute v_{ph} and v_{gr} in $\text{km}\,\text{s}^{-1}$.

5. A string of length $L = 1$ m is plucked one-quarter of the way from one end (at $x = L/4$). (a) Sketch the initial triangular displacement. (b) Write down the Fourier sine series for this initial shape by evaluating the integral for b_n. (c) Which harmonics have zero amplitude, and why? (d) Compare qualitatively with the midpoint-pluck case discussed in the lecture: which produces more high-harmonic content, and why does the plucking position affect the timbre?

6. (*Connecting to the cosmos.*) The cosmic microwave background has temperature $T_0 = 2.725$ K today. At recombination ($z_d \approx 1050$), the temperature was $T_d = T_0(1 + z_d) \approx 2860$ K. (a) For *pure radiation*, the sound speed is $c_s = c/\sqrt{3}$. By what factor does this exceed the speed of sound in air at 20°C? (b) Explain in physical terms—referring to $c_s = \sqrt{B/\rho}$—why the sound speed in the pre-recombination plasma is so extraordinarily large. What is the analogue of the bulk modulus B, and what is the analogue of the density ρ? (c) In Lecture 4 we will show that adding baryons reduces c_s to $c/\sqrt{3(1 + R_b)}$ with $R_b \approx 0.6$ near recombination. Compute the corrected c_s and express it as a fraction of c. (d) The sound horizon is $r_s = c_s \times t_d$, where $t_d \approx 380{,}000$ years $= 1.2 \times 10^{13}$ s is the age of the universe at recombination. Using the corrected c_s from part (c), estimate r_s in Mpc (1 Mpc $= 3.086 \times 10^{22}$ m). Compare with the measured value of 147 Mpc.