What did Max Planck do?

Tayur Lectures on Physics

Contents

1 Introduction: The Crisis in Classical Physics

By the late 19th century, classical physics appeared to be a complete and triumphant theory. Newton's mechanics explained motion, Maxwell's equations unified electricity and magnetism, and thermodynamics described heat and energy. However, when physicists turned their attention to blackbody radiation—the electromagnetic radiation emitted by heated objects—classical theory made a prediction that was spectacularly wrong.

The problem became known as the **ultraviolet catastrophe**, and its resolution by Max Planck in 1900 marked the birth of quantum mechanics and revolutionized our understanding of nature at the atomic scale.

2 What is Blackbody Radiation?

A **blackbody** is an idealized object that:

1. Absorbs all electromagnetic radiation incident upon it (no reflection)
2. Emits radiation with a spectrum that depends only on its temperature T

A good approximation is a small hole in a cavity. Radiation entering the hole is trapped inside and absorbed through multiple reflections. The radiation escaping through the hole represents the emission spectrum.

The key experimental observation was that the ***spectral energy density*** $u(\nu, T)$ (energy per unit volume per unit frequency) has a characteristic shape: it rises from zero at low frequencies, reaches a maximum, and then falls back to zero at high frequencies. Crucially, this shape depends only on temperature, not on the material properties of the blackbody.

3 The Classical (Rayleigh-Jeans) Derivation

Lord Rayleigh (and later James Jeans with corrections) attempted to derive the blackbody spectrum using classical physics. Their approach combined electromagnetic theory with statistical mechanics.

3.1 Step 1: Counting Electromagnetic Modes

Consider electromagnetic radiation inside a cubic cavity of side length L. The electromagnetic waves must satisfy boundary conditions: the electric field must vanish at the metallic walls. This constraint means only certain wavelengths (modes) are allowed, similar to standing waves on a string.

For a 3D cavity, the allowed wavelengths correspond to wave vectors:

$$\vec{k} = \left(\frac{n_x \pi}{L}, \frac{n_y \pi}{L}, \frac{n_z \pi}{L} \right) \tag{1}$$

where n_x, n_y, n_z are positive integers (1, 2, 3, . . .).

The magnitude of the wave vector is related to frequency by:

$$|\vec{k}| = k = \frac{2\pi\nu}{c} \tag{2}$$

where c is the speed of light and ν is the frequency.

To count modes with frequencies between ν and $\nu + d\nu$, we work in k-space. Each mode occupies a volume of $(\pi/L)^3$ in k-space. The number of modes with wave vector magnitude between k and $k + dk$ is the volume of a spherical shell (considering only positive octant since $n_x, n_y, n_z > 0$):

$$\text{Number of modes} = \frac{1}{8} \cdot \frac{4\pi k^2 dk}{(\pi/L)^3} \tag{3}$$

Simplifying:

$$= \frac{L^3 k^2 dk}{2\pi^2} \tag{4}$$

Since $k = 2\pi\nu/c$, we have $dk = (2\pi/c)d\nu$. Substituting:

$$\text{Number of modes} = \frac{L^3}{2\pi^2} \cdot \left(\frac{2\pi\nu}{c} \right)^2 \cdot \frac{2\pi}{c} d\nu = \frac{4\pi L^3 \nu^2}{c^3} d\nu \tag{5}$$

However, electromagnetic waves have **two independent polarization states** for each mode (think of horizontal and vertical polarizations). Therefore:

$$g(\nu)d\nu = 2 \times \frac{4\pi L^3 \nu^2}{c^3} d\nu = \frac{8\pi L^3 \nu^2}{c^3} d\nu \tag{6}$$

This is the number of electromagnetic modes in the frequency range $[\nu, \nu + d\nu]$ in a cavity of volume $V = L^3$. The **mode density** per unit volume is:

$$\boxed{\rho(\nu)d\nu = \frac{g(\nu)d\nu}{V} = \frac{8\pi\nu^2}{c^3} d\nu} \tag{7}$$

3.2 Step 2: Energy Per Mode (Equipartition Theorem)

Classical statistical mechanics provides a powerful result called the **equipartition theorem**: at thermal equilibrium at temperature T, each quadratic degree of freedom has an average energy of $(1/2)k_BT$, where k_B is Boltzmann's constant.

An electromagnetic mode is a harmonic oscillator with both kinetic energy (from oscillating electric and magnetic fields) and potential energy. The total energy has two quadratic terms, giving:

$$\boxed{\langle E \rangle = 2 \times \frac{1}{2} k_B T = k_B T} \tag{8}$$

This is the average energy per mode at temperature T.

3.3 Step 3: The Rayleigh-Jeans Formula

The spectral energy density $u(\nu, T)$ is the energy per unit volume per unit frequency. We obtain it by multiplying the mode density by the average energy per mode:

$$u(\nu, T) = \rho(\nu) \times \langle E \rangle \tag{9}$$

$$\boxed{u(\nu, T) = \frac{8\pi\nu^2}{c^3} \cdot k_B T} \tag{10}$$

This is the Rayleigh-Jeans Law.

3.4 The Ultraviolet Catastrophe

The Rayleigh-Jeans formula has a fatal flaw. Notice that $u(\nu, T)$ is proportional to ν^2. As frequency increases:

$$\lim_{\nu\to\infty} u(\nu, T) = \lim_{\nu\to\infty} \frac{8\pi\nu^2 k_B T}{c^3} \to \infty \tag{11}$$

The energy density diverges at high frequencies! This means:

The total energy in the cavity would be:

$$U = V \int_0^\infty u(\nu, T) d\nu = V \int_0^\infty \frac{8\pi\nu^2 k_B T}{c^3} d\nu \to \infty \tag{12}$$

This is physically absurd! It predicts that:

1. Any blackbody at any temperature should emit infinite energy

2. All energy should rapidly flow into high-frequency (ultraviolet and beyond) modes

3. Every object should glow with infinite intensity in ultraviolet light

This catastrophic failure became known as the **ultraviolet catastrophe**. The Rayleigh-Jeans formula does work well at low frequencies (long wavelengths) where experiments matched predictions, but fails spectacularly at high frequencies.

Classical physics had failed.

4 Planck's Revolutionary Solution

Max Planck, working at the University of Berlin, approached the problem differently. Rather than starting from first principles, he worked backward from experimental data to find a formula that fit, then sought a theoretical justification.

4.1 Planck's Empirical Formula

On October 19, 1900, Planck proposed an empirical formula that fit the experimental data perfectly:

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_B T} - 1} \tag{13}$$

where h is a new constant (now called Planck's constant). This formula:

- Matches experimental data at all frequencies and temperatures
- Reduces to the Rayleigh-Jeans law at low frequencies
- Goes to zero (not infinity!) at high frequencies

But *why* did this formula work? Planck spent two months searching for a theoretical derivation.

4.2 Planck's Derivation: The Quantum Hypothesis

On December 14, 1900, Planck presented his theoretical derivation to the German Physical Society. The key was a radical new assumption that seemed desperate at the time but proved revolutionary:

Energy can only be exchanged in discrete packets (quanta).

Specifically, Planck assumed that an electromagnetic oscillator of frequency ν can only have energies that are integer multiples of $h\nu$:

$$E_n = nh\nu \quad \text{where } n = 0, 1, 2, 3, \ldots \tag{14}$$

This was completely contrary to classical physics, where energy is continuous. Let's follow Planck's derivation step by step.

4.2.1 Step 1: Boltzmann's Entropy Formula

Planck used Ludwig Boltzmann's statistical interpretation of entropy. Boltzmann had shown that entropy S is related to the number of microstates W by:

$$S = k_B \ln W \tag{15}$$

where W is the number of ways to arrange a system with a given macroscopic state (total energy). At thermal equilibrium, the system maximizes entropy subject to the constraint of fixed total energy.

4.2.2 Step 2: Setting Up the Problem

Consider N identical oscillators (electromagnetic modes) of frequency ν, with total energy U_N. We want to find the most probable distribution of energy among these oscillators.

Using Planck's quantum hypothesis, each oscillator can only have energies $0, h\nu, 2h\nu, 3h\nu, \ldots$ If the total energy is U_N, and each quantum has energy $\varepsilon = h\nu$, then the total number of quanta is:

$$P = \frac{U_N}{\varepsilon} = \frac{U_N}{h\nu} \tag{16}$$

The question becomes: In how many ways can we distribute P indistinguishable quanta among N distinguishable oscillators?

4.2.3 Step 3: Combinatorics - Counting Microstates

This is a classic "stars and bars" combinatorics problem. Imagine P quanta (stars) and $N - 1$ dividers (bars) separating them into N groups (oscillators). The total number of arrangements is:

$$W = \frac{(P + N - 1)!}{P! \cdot (N - 1)!} \tag{17}$$

For example, with $P = 3$ quanta and $N = 2$ oscillators:

$$\begin{array}{ll}
\bigstar\bigstar\bigstar| & \text{(oscillator 1 has 3, oscillator 2 has 0)} \\
\bigstar\bigstar|\bigstar & \text{(oscillator 1 has 2, oscillator 2 has 1)} \\
\bigstar|\bigstar\bigstar & \text{(oscillator 1 has 1, oscillator 2 has 2)} \\
|\bigstar\bigstar\bigstar & \text{(oscillator 1 has 0, oscillator 2 has 3)}
\end{array}$$

There are $(3 + 2 - 1)!/(3! \times 1!) = 4!/(3! \times 1!) = 4$ ways, as shown.

4.2.4 Step 4: Stirling's Approximation

For large N and P, we need to evaluate $\ln W$. We use **Stirling's approximation**:

$$\ln(n!) \approx n \ln n - n \quad \text{(for large } n\text{)} \tag{18}$$

Applying this to our expression:

$$\ln W = \ln[(P + N - 1)!] - \ln(P!) - \ln[(N - 1)!] \tag{19}$$

Using Stirling's approximation (and assuming $N \gg 1$ so $N - 1 \approx N$):

$$\ln W \approx (P + N) \ln(P + N) - (P + N) - P \ln P + P - N \ln N + N \tag{20}$$

$$= (P + N) \ln(P + N) - P \ln P - N \ln N \tag{21}$$

We can rewrite this as:

$$\ln W = (P + N) \ln(P + N) - P \ln P - N \ln N \tag{22}$$

4.2.5 Step 5: Entropy and the Average Energy

The entropy is:

$$S = k_B \ln W \tag{23}$$

In thermodynamics, entropy and energy are related by:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U_N}\right)_N \tag{24}$$

Since $P = U_N/(h\nu)$, we have $\partial P/\partial U_N = 1/(h\nu)$. Therefore:

$$\frac{1}{T} = \frac{\partial S}{\partial P} \cdot \frac{\partial P}{\partial U_N} = \frac{k_B}{h\nu} \cdot \frac{\partial \ln W}{\partial P} \tag{25}$$

Now we calculate $\partial \ln W/\partial P$. From our expression:

$$\ln W = (P + N)\ln(P + N) - P\ln P - N\ln N \tag{26}$$

Taking the derivative with respect to P:

$$\frac{\partial \ln W}{\partial P} = \ln(P + N) + \frac{P + N}{P + N} - \ln P - \frac{P}{P} \tag{27}$$

$$= \ln(P + N) + 1 - \ln P - 1 = \ln\left[\frac{P + N}{P}\right] \tag{28}$$

Therefore:

$$\frac{1}{T} = \frac{k_B}{h\nu} \ln\left[\frac{P + N}{P}\right] = \frac{k_B}{h\nu} \ln\left[1 + \frac{N}{P}\right] \tag{29}$$

Solving for P:

$$\ln\left[1 + \frac{N}{P}\right] = \frac{h\nu}{k_B T} \tag{30}$$

$$1 + \frac{N}{P} = e^{h\nu/k_B T} \tag{31}$$

$$\frac{N}{P} = e^{h\nu/k_B T} - 1 \tag{32}$$

$$\frac{P}{N} = \frac{1}{e^{h\nu/k_B T} - 1} \tag{33}$$

4.2.6 Step 6: The Average Energy Per Oscillator

The average energy per oscillator is:

$$\langle E \rangle = \frac{U_N}{N} = \frac{P \times h\nu}{N} = h\nu \cdot \frac{P}{N} \tag{34}$$

Substituting our result for P/N:

$$\boxed{\langle E \rangle = \frac{h\nu}{e^{h\nu/k_B T} - 1}} \tag{35}$$

This is Planck's quantum result for the average energy per oscillator!

4.3 Planck's Blackbody Formula

Just as in the classical derivation, the spectral energy density is:

$$u(\nu, T) = \rho(\nu) \times \langle E \rangle \tag{36}$$

The mode density is unchanged:

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3} \tag{37}$$

But the average energy per mode is now Planck's quantum expression:

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/k_BT} - 1} \tag{38}$$

Therefore:

$$\boxed{u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_BT} - 1}} \tag{39}$$

Planck's Law

5 Analysis of Planck's Formula

5.1 Comparison with Classical Result

The crucial difference is in the average energy per mode:

Classical (Rayleigh-Jeans): $\langle E \rangle = k_BT$

Quantum (Planck): $\langle E \rangle = \frac{h\nu}{e^{h\nu/k_BT} - 1}$

5.2 Low Frequency Limit

For low frequencies (or high temperatures), $h\nu \ll k_BT$. We can expand the exponential:

$$e^{h\nu/k_BT} \approx 1 + \frac{h\nu}{k_BT} + \cdots \tag{40}$$

Therefore:

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/k_BT} - 1} \approx \frac{h\nu}{h\nu/(k_BT)} = k_BT \tag{41}$$

Planck's formula reduces to the classical result at low frequencies! This is why the Rayleigh-Jeans law worked for long wavelengths.

5.3 High Frequency Limit - No Catastrophe!

For high frequencies (or low temperatures), $h\nu \gg k_BT$. Then:

$$e^{h\nu/k_BT} \gg 1 \tag{42}$$

So:

$$\langle E \rangle \approx \frac{h\nu}{e^{h\nu/k_BT}} = h\nu \cdot e^{-h\nu/k_BT} \tag{43}$$

And the spectral energy density:

$$u(\nu, T) \approx \frac{8\pi h\nu^3}{c^3} \cdot e^{-h\nu/k_BT} \tag{44}$$

The exponential decay makes $u(\nu, T) \to 0$ as $\nu \to \infty$

The ultraviolet catastrophe is averted! The key is that at high frequencies, the quantum of energy $h\nu$ becomes so large that thermal fluctuations (with average energy $\sim k_BT$) cannot excite these modes. High-frequency modes are effectively "frozen out" and carry negligible energy.

5.4 Total Energy is Finite

The total energy density is now:

$$u_{\text{total}} = \int_0^\infty u(\nu, T) d\nu = \int_0^\infty \frac{8\pi h \nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_B T} - 1} d\nu \tag{45}$$

This integral converges (equals $\frac{8\pi^5 k_B^4}{15c^3h^3}T^4$), giving the Stefan-Boltzmann law with the correct coefficient. The total energy is proportional to T^4, not infinite!

6 Historical Significance and Conclusion

Planck's resolution of the ultraviolet catastrophe marked a pivotal moment in physics:

1. **Birth of Quantum Mechanics:** The quantization hypothesis $E = nh\nu$ was the first quantum concept, though Planck himself was uncomfortable with its radical implications.

2. **Planck's Constant:** The constant $h \approx 6.626 \times 10^{-34}$ J·s became one of the fundamental constants of nature, setting the scale for quantum effects.

3. **Challenge to Classical Physics:** Planck showed that classical physics fails at the atomic scale, opening the door for Einstein's photon hypothesis (1905), Bohr's atomic model (1913), and eventually the full quantum theory of Heisenberg and Schrödinger (1925-1926).

4. **Practical Applications:** Understanding blackbody radiation enabled precise temperature measurements (pyrometry), laid foundations for understanding stellar spectra, and continues to be essential in modern physics and astronomy.

Ironically, Planck regarded quantization as a "mathematical trick" and spent years trying to reconcile it with classical physics. He received the Nobel Prize in 1918, but the full meaning of his discovery—that nature itself is fundamentally discontinuous at small scales—only became clear through the work of subsequent physicists.

The ultraviolet catastrophe and its resolution remind us that progress in physics often comes from carefully examining where theory and experiment disagree. What seemed like a technical problem about cavity radiation turned out to be a window into a revolutionary new understanding of reality itself.

Key Formulas Summary

Mode density:

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3}$$

Classical average energy per mode:

$$\langle E \rangle_{\text{classical}} = k_B T$$

Rayleigh-Jeans Law (classical):

$$u(\nu, T) = \frac{8\pi\nu^2 k_B T}{c^3} \quad \to \quad \infty \text{ as } \nu \to \infty$$

Planck's quantum hypothesis:

$$E_n = nh\nu, \quad n = 0, 1, 2, 3, \ldots$$

Planck's average energy per mode:

$$\langle E\rangle_{\text{Planck}} = \frac{h\nu}{e^{h\nu/k_B T} - 1}$$

Planck's Law:

$$u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_B T} - 1}$$

Where: $h = 6.626 \times 10^{-34}$ *J·s (Planck's constant),* $k_B = 1.381 \times 10^{-23}$ *J/K (Boltzmann's constant),* $c = 3.00 \times 10^8$ *m/s (speed of light)*