The Chandrasekhar Limit

Tayur Lectures on Physics

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1 Introduction

Subrahmanyan Chandrasekhar (1910–1995), known simply as "Chandra" to his colleagues, was one of the most influential astrophysicists of the twentieth century. Born in Lahore (then British India, now Pakistan), he made his most famous discovery during a voyage from India to England in 1930, when he was just nineteen years old. This discovery—the existence of a maximum mass for white dwarf stars—would revolutionize our understanding of stellar evolution and earn him the Nobel Prize in Physics in 1983.

The young Chandrasekhar's calculation combined two of the great pillars of early twentieth-century physics: Einstein's special relativity and the newly developed quantum mechanics. What he found was both elegant and unsettling: stars above a certain mass could not end their lives as white dwarfs. This seemingly abstract result would eventually lead to our understanding of neutron stars and black holes, and would provide astronomers with one of their most reliable distance measurement tools.

2 The Physical Context: Why Do Stars Need Support?

Before we can appreciate Chandrasekhar's discovery, we need to understand the fundamental problem facing any star: gravity is relentless. A star is essentially a massive ball of gas, and every bit of that gas pulls on every other bit through gravitational attraction.

2.1 Hydrostatic Equilibrium

For a star to maintain a stable radius, there must be an outward pressure that balances the inward pull of gravity. This condition is called *hydrostatic equilibrium*. Mathematically, at any radius r within a star, we can write:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{1}$$

where P is the pressure, G is Newton's gravitational constant, $M(r)$ is the mass enclosed within radius r, and $\rho(r)$ is the density at radius r. The negative sign indicates that pressure must increase as we go inward (toward smaller r) to support the weight of the overlying material.

For most of a star's life, this pressure comes from the thermal motion of particles—hot gas pushes outward. The energy to keep the gas hot comes from nuclear fusion in the star's core. But what happens when the fuel runs out?

2.2 The White Dwarf Endpoint

When a star like our Sun exhausts its nuclear fuel, it can no longer maintain the high temperatures needed for thermal pressure support. The core collapses until it reaches an extraordinary density—roughly a million times the density of water. At this point, a new kind of pressure takes over: *degeneracy pressure*.

To understand degeneracy pressure, we turn to quantum mechanics.

3 Quantum Mechanics: The Pauli Exclusion Principle

In 1925, Wolfgang Pauli articulated a fundamental principle of quantum mechanics: no two fermions (particles with half-integer spin, like electrons) can occupy the same quantum state simultaneously. Electrons, being fermions with spin-1/2, obey this exclusion principle.

3.1 Consequences for Dense Matter

In ordinary matter, electrons occupy the lowest available energy states, with the Pauli principle ensuring that each electron has a unique quantum state. As matter is compressed, the number of available states in a given volume decreases, forcing electrons into higher and higher energy states.

This creates a pressure even at zero temperature, because the electrons resist being squeezed into already-occupied states. This is *electron degeneracy pressure*, and it depends only on density, not temperature.

4 The Non-Relativistic Degenerate Electron Gas

Let us now develop the physics quantitatively, starting with the case where electron speeds are much less than the speed of light c.

4.1 The Fermi Momentum

Consider a volume V containing N electrons. In momentum space, quantum mechanics tells us that each quantum state occupies a volume of h^3 (where h is Planck's constant). Taking into account electron spin (two spin states per spatial state), the total number of available states up to momentum p is:

$$N = 2 \cdot \frac{4\pi p_F^3}{3} \cdot \frac{V}{h^3} \tag{2}$$

where p_F is called the *Fermi momentum*—the momentum of the highest-energy occupied state. Solving for p_F:

$$p_F = \left(\frac{3Nh^3}{8\pi V}\right)^{1/3} = \left(\frac{3h^3}{8\pi}\right)^{1/3} n_e^{1/3} \tag{3}$$

where $n_e = N/V$ is the number density of electrons.

4.2 The Fermi Energy

In the non-relativistic limit, the kinetic energy of an electron with momentum p is:

$$E = \frac{p^2}{2m_e} \tag{4}$$

where m_e is the electron mass. The Fermi energy E_F (the energy of the most energetic electron) is therefore:

$$E_F = \frac{p_F^2}{2m_e} = \frac{1}{2m_e}\left(\frac{3h^3}{8\pi}\right)^{2/3} n_e^{2/3} \tag{5}$$

4.3 Degeneracy Pressure

The pressure exerted by this degenerate electron gas can be derived from statistical mechanics. The result is:

$$P_{\text{deg}} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m_e} n_e^{5/3} \tag{6}$$

where $\hbar = h/(2\pi)$ is the reduced Planck constant. The key feature here is the $n_e^{5/3}$ dependence: pressure increases with density, but with an exponent less than 2.

4.4 Connection to Mass Density

In a white dwarf, the density is so high that matter is completely ionized. The mass density ρ is approximately:

$$\rho \approx \mu_e m_p n_e \tag{7}$$

where m_p is the proton mass and μ_e is the mean molecular weight per electron. For a star made of carbon and oxygen (typical for white dwarfs), we have roughly 6 protons and neutrons per 6 electrons, so $\mu_e \approx 2$.

Expressing the pressure in terms of mass density:

$$P_{\text{deg}} = K_{\text{NR}} \rho^{5/3} \tag{8}$$

where

$$K_{\text{NR}} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m_e(\mu_e m_p)^{5/3}} \tag{9}$$

This is the equation of state for a non-relativistic degenerate electron gas.

5 The Mass-Radius Relation for Non-Relativistic White Dwarfs

With the equation of state in hand, we can now relate the mass and radius of a white dwarf.

5.1 Scaling Arguments

For a star of mass M and radius R, the typical density is:

$$\rho \sim \frac{M}{R^3} \tag{10}$$

The degeneracy pressure scales as:

$$P \sim K_{\text{NR}}\rho^{5/3} \sim K_{\text{NR}}\frac{M^{5/3}}{R^5} \tag{11}$$

The gravitational pressure (the pressure needed to support the star's weight) scales as:

$$P_{\text{grav}} \sim \frac{GM^2}{R^4} \tag{12}$$

For hydrostatic equilibrium, we require $P \sim P_{\text{grav}}$:

$$K_{\text{NR}}\frac{M^{5/3}}{R^5} \sim \frac{GM^2}{R^4} \tag{13}$$

Solving for R:

$$R \sim \frac{K_{\text{NR}}}{GM^{1/3}} \tag{14}$$

This remarkable result shows that *more massive white dwarfs are smaller*! The degeneracy pressure, with its $\rho^{5/3}$ dependence, allows this inverse relationship between mass and radius.

6 The Relativistic Regime: Chandrasekhar's Insight

The preceding analysis assumes that electrons move slowly compared to the speed of light. But as we increase the mass (and therefore the density) of a white dwarf, the Fermi momentum p_F increases, and eventually electron speeds approach c. This is where Chandrasekhar's crucial contribution enters.

6.1 Relativistic Kinematics

When particle speeds approach c, we must use Einstein's relativistic energy-momentum relation:

$$E^2 = (pc)^2 + (m_e c^2)^2 \tag{15}$$

In the *ultra-relativistic limit*, where $pc \gg m_e c^2$, this simplifies to:

$$E \approx pc \tag{16}$$

6.2 The Ultra-Relativistic Equation of State

Following similar arguments as before, but now with $E = pc$, the pressure of an ultra-relativistic degenerate electron gas is:

$$P_{\text{rel}} = K_{\text{UR}} \rho^{4/3} \tag{17}$$

where

$$K_{\text{UR}} = \frac{(3\pi^2)^{1/3} \hbar c}{4(\mu_e m_p)^{4/3}} \tag{18}$$

The crucial difference is the exponent: pressure now scales as $\rho^{4/3}$ instead of $\rho^{5/3}$.

7 The Chandrasekhar Limit

7.1 The Critical Scaling

Now let us repeat our scaling analysis with the relativistic equation of state. The degeneracy pressure is:

$$P \sim K_{\text{UR}} \rho^{4/3} \sim K_{\text{UR}} \frac{M^{4/3}}{R^4} \tag{19}$$

The gravitational pressure still scales as:

$$P_{\text{grav}} \sim \frac{GM^2}{R^4} \tag{20}$$

Setting these equal:

$$K_{\text{UR}} \frac{M^{4/3}}{R^4} \sim \frac{GM^2}{R^4} \tag{21}$$

The R^4 terms cancel, leaving:

$$K_{\text{UR}} M^{4/3} \sim GM^2 \tag{22}$$

Solving for M:

$$M \sim \left(\frac{K_{\mathrm{UR}}}{G}\right)^{3/2} \tag{23}$$

This is the stunning result: in the ultra-relativistic regime, there is a *unique mass* for which degeneracy pressure can balance gravity. There is no stable radius—either the star is at this critical mass, or degeneracy pressure cannot support it.

7.2 The Chandrasekhar Mass

A detailed calculation, properly integrating the equations of stellar structure (rather than using scaling arguments), gives the precise value of this limiting mass. Chandrasekhar found:

$$M_{\mathrm{Ch}} = \frac{\omega_3^0 \sqrt{3\pi}}{2} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{(\mu_e m_p)^2} \tag{24}$$

where $\omega_3^0 \approx 2.018$ is a dimensionless constant that comes from solving the Lane-Emden equation (the precise form of the stellar structure equations for a polytropic equation of state).

Evaluating this numerically with $\mu_e = 2$ (appropriate for carbon-oxygen white dwarfs):

$$M_{\mathrm{Ch}} \approx 1.44\, M_\odot \tag{25}$$

where $M_\odot$ is the mass of the Sun. This is the *Chandrasekhar limit*.

7.3 Physical Interpretation

The Chandrasekhar limit has a profound physical meaning: **no white dwarf can have a mass greater than about 1.44 solar masses**. If you try to add mass to a white dwarf approaching this limit, the extra compression pushes the electrons into the ultra-relativistic regime, where the equation of state changes from $P \propto \rho^{5/3}$ to $P \propto \rho^{4/3}$. At exactly the Chandrasekhar mass, the support mechanism fails—degeneracy pressure can no longer balance gravity.

Stars more massive than this limit cannot end their lives as white dwarfs. They must collapse further, becoming either neutron stars (supported by neutron degeneracy pressure) or, if massive enough, black holes.

8 Historical Context and Reception

8.1 The 1930 Voyage

Chandrasekhar made his famous calculation while traveling by ship from India to Cambridge, England, in the summer of 1930. He had just completed his undergraduate degree in Madras and was heading to study with Ralph Fowler, who had pioneered the application of quantum mechanics to white dwarfs.

During the voyage, Chandrasekhar read a paper by Fowler on degenerate matter and began to wonder what would happen at very high densities, where relativistic effects become important. By the time he arrived in England, he had worked out the basic result: there was a maximum mass for white dwarfs.

8.2 Eddington's Opposition

When Chandrasekhar presented his work at a Royal Astronomical Society meeting in 1935, he faced fierce criticism from Arthur Eddington, one of the most prominent astrophysicists of the era. Eddington argued that the result was absurd—that "various accidents" would prevent a star from catastrophically collapsing. In Eddington's words, there should be "a law of Nature to prevent a star from behaving in this absurd way!"

Eddington's opposition was based partly on physical intuition (he couldn't accept that stars could collapse indefinitely) and partly on a misunderstanding of the physics. He believed that high-energy electrons would lose energy through radiation before reaching the relativistic regime. However, Chandrasekhar's calculation was correct: degeneracy pressure depends on density, not temperature, and there is no radiation loss in a degenerate gas at zero temperature.

8.3 Eventual Acceptance

Despite Eddington's prestige, other physicists gradually recognized the validity of Chandrasekhar's work. By the 1960s, with the discovery of neutron stars and growing evidence for black holes, the Chandrasekhar limit was accepted as fundamental to our understanding of stellar evolution. Chandrasekhar was awarded the Nobel Prize in 1983 "for his theoretical studies of the physical processes of importance to the structure and evolution of the stars."

9 Type Ia Supernovae and Cosmology

The Chandrasekhar limit is not just an abstract piece of theoretical physics—it has become central to observational cosmology through its connection to Type Ia supernovae.

9.1 The Mechanism of Type Ia Supernovae

A Type Ia supernova occurs when a white dwarf in a binary system accretes matter from a companion star. As the white dwarf's mass approaches the Chandrasekhar limit, the increased density and temperature in the core trigger runaway carbon fusion.

The key insight is that the white dwarf is supported by electron degeneracy pressure, which is *temperature-independent*. Unlike a normal star, where increased temperature causes expansion and self-regulation, a white dwarf cannot expand to cool itself. When carbon fusion ignites, the temperature rises rapidly, increasing the fusion rate in a runaway process. Within seconds, the entire star explodes.

Crucially, this explosion always happens at approximately the same mass—the Chandrasekhar limit. Since the mass is nearly constant, and the physics of the explosion is the same each time, Type Ia supernovae have remarkably uniform peak luminosities.

9.2 Standard Candles

A *standard candle* is an astronomical object of known intrinsic brightness. If you know how bright something truly is (its absolute magnitude), and you measure how bright it appears from Earth (its apparent magnitude), you can calculate its distance using the inverse-square law:

$$\frac{L_{\text{observed}}}{L_{\text{intrinsic}}} = \frac{1}{4\pi d^2} \tag{26}$$

where L represents luminosity and d is the distance.

Type Ia supernovae, with their uniform peak luminosities set by the Chandrasekhar limit, are excellent standard candles. Astronomers have refined this further: there are small variations in peak brightness, but these correlate with the rate at which the supernova fades. By measuring the light curve (brightness versus time), astronomers can determine the exact peak luminosity, making Type Ia supernovae *standardizable candles*.

9.3 The Accelerating Universe

In the late 1990s, two independent teams used Type Ia supernovae to measure distances to far-away galaxies. By comparing these distances with the galaxies' redshifts (which indicate how fast they're receding), the teams could map out the expansion history of the universe.

The results were shocking: distant supernovae were fainter than expected, implying they were farther away than predicted by models of a decelerating universe. The universe's expansion was not slowing down under gravity's pull—it was *speeding up*.

This discovery of cosmic acceleration led to the inference of *dark energy*, a mysterious component that makes up about 68% of the universe's energy density. The leaders of the two teams (Saul Perlmutter, Brian Schmidt, and Adam Riess) were awarded the Nobel Prize in Physics in 2011.

None of this would have been possible without reliable distance measurements, which in turn depend on the remarkable uniformity of Type Ia supernovae—a uniformity guaranteed by the Chandrasekhar limit.

10 Modern Refinements and Open Questions

10.1 Variations in μ_e

The precise value of the Chandrasekhar limit depends on the composition of the white dwarf through the parameter μ_e. For a carbon-oxygen white dwarf, $\mu_e \approx 2$ and $M_{\text{Ch}} \approx 1.44 M_\odot$. However, for heavier elements:

- Pure iron: $\mu_e \approx 2.15$, giving $M_{\text{Ch}} \approx 1.26 M_\odot$

- Helium: $\mu_e = 2$, giving $M_{\rm Ch} \approx 1.44 M_\odot$

10.2 Rotation and Magnetic Fields

Chandrasekhar's original calculation assumed a non-rotating, non-magnetic white dwarf. Rapid rotation can provide additional centrifugal support, allowing white dwarfs to exceed the classical Chandrasekhar limit by perhaps 10–20%. Similarly, extremely strong magnetic fields (found in some white dwarfs called magnetic white dwarfs) can modify the equation of state and change the limit slightly.

10.3 Super-Chandrasekhar Supernovae?

In recent years, astronomers have observed a few Type Ia supernovae that appear to come from progenitors more massive than $1.44 M_\odot$. These "super-Chandrasekhar" events are controversial and not yet fully understood. Possible explanations include:

- Rapidly rotating white dwarfs
- Strong magnetic fields
- Mergers of two white dwarfs
- Errors in luminosity estimation

This remains an active area of research.

11 Broader Impact: From Stars to Quantum Matter

Chandrasekhar's work on white dwarfs exemplifies the deep connections between different areas of physics. His calculation required:

- **Quantum mechanics**: the Pauli exclusion principle and the properties of degenerate matter
- **Special relativity**: the correct energy-momentum relation for high-speed particles
- **General relativity** (implicitly): Newton's gravity is used, but in the most extreme cases, general relativistic corrections become important
- **Statistical mechanics**: relating microscopic properties of particles to macroscopic properties like pressure
- **Stellar structure theory**: solving the equations of hydrostatic equilibrium

The Chandrasekhar limit also provides a beautiful example of dimensional analysis. The combination $(\hbar c/G)^{1/2}$ has units of mass and equals the *Planck mass*, $m_{\rm Pl} \approx 2.2 \times 10^{-8}$ kg. The Chandrasekhar mass is essentially:

$$M_{\rm Ch} \sim \left(\frac{m_{\rm Pl}}{m_p}\right)^2 m_p \sim 10^{38} m_p \sim M_\odot \tag{27}$$

This shows that the Chandrasekhar limit is set by the interplay of quantum mechanics ($\hbar$), relativity (c), gravity (G), and nuclear physics (m_p). It is a genuinely fundamental scale in astrophysics.

12 Conclusion

Subrahmanyan Chandrasekhar's determination of the maximum mass for white dwarf stars stands as one of the great achievements of twentieth-century astrophysics. Working from first principles—quantum mechanics, special relativity, and stellar structure theory—he showed that stars above approximately 1.44 solar masses cannot end their lives as white dwarfs.

This result was initially controversial, opposed by giants like Arthur Eddington. But it has stood the test of time and observation. The Chandrasekhar limit is now central to our understanding of stellar evolution, supernova explosions, and cosmic distance measurements. Type Ia supernovae, whose uniform luminosities derive directly from the Chandrasekhar limit, have become one of astronomy's most important tools, leading to the discovery of cosmic acceleration and dark energy.

Chandrasekhar's work reminds us that theoretical physics, pursued with rigor and imagination, can reveal deep truths about the universe—truths that may not be immediately accepted, but which ultimately reshape our understanding of the cosmos.

Key Equations Summary

$$\text{Non-relativistic degeneracy pressure:} \quad P \propto \rho^{5/3} \tag{28}$$

$$\text{Ultra-relativistic degeneracy pressure:} \quad P \propto \rho^{4/3} \tag{29}$$

$$\text{Chandrasekhar limit:} \quad M_{\rm Ch} \approx 1.44 M_\odot \quad (\mu_e = 2) \tag{30}$$

$$\text{General formula:} \quad M_{\rm Ch} \propto \frac{1}{\mu_e^2}\left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2} \tag{31}$$

Further Reading

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